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Gurukul Kangri Vishwavidyalaya, Hardwar

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पेज फटे हैं 1-16 तक

$$\frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \left(- \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} \right) + B(v-u) \qquad \ldots (5)$$

$$\frac{\partial v}{\partial t} = \frac{u-v}{t} \qquad \ldots (6)$$

where $B = \frac{M}{\tau} = \frac{N_0 K a^2}{\mu}$, $M = \underline{No\,m}$

$$\tau = \frac{v\,m}{k a^2} \;;$$

m being the mass of red blood cells, the radius

$$\beta = 1 - \alpha\left(1 + \cos\frac{\pi z}{z_0}\right) \qquad \ldots (7)$$

where, $\beta = \frac{R}{a}$, $\alpha = \frac{\delta_0}{a}$

and the pressure gradient may be assumed in the form

$$\frac{\partial p}{\partial x} = C_1 F(t) = e^{-\lambda t} \text{(say)} \qquad \ldots (8)$$

where, C_1 is a constant.

Equation (6) may be solved for v, using the particle initial condition $v(r,0) = 0$, to give

$$v(r,t) = \frac{1}{\tau} e^{-t/\tau} u(r,\eta) e^{\eta/\tau} d \qquad \ldots (9)$$

On substituting (8) and (9) in equation (5) we get

$$\frac{\partial u}{\partial t} = C_1 F(t) + \left(\frac{\partial^2 u}{\partial t^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right) + B\left[\frac{1}{\tau} e^{t/\tau} \int_0^{\tau} u(r,\eta) e^{\eta/\tau} d\eta - u\right] \qquad \ldots (10)$$

Once the above fluid momentum, equation is solved for u with the help of equation (3) for radius varying with axial distance, v can be obtained from equation (9).

3. SOLUTION OF THE PROBLEM :

Applying the Laplace transform in equation (5), we get

$$\frac{\partial^2 \bar{u}}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{u}}{\partial r} - \left(\frac{s^2 - BK_2}{s}\right)\bar{u} + c_1\bar{F}(s) = 0 \qquad \ldots (11)$$

where $K_2 = \frac{1}{\tau}$

Solving the equation (11) we get complimentary function as follows.

$$\bar{u} = aJ_0(Qr) + BY_0(Qr) \qquad \ldots (12)$$

where $Q = \left(\frac{s^2 + Bs - BK_2}{s}\right)^{1/2}$

J_0 and Y_0 are modified Bessel's functions.

Variation of the parameters method has been used to evaluate second order differential equation. We have obtained the solution for $\bar{u}$, using equation (12) as assumed solutions.

$\bar{u} = AJ_0\ (Qr) + BY_0\ (Qr)$

where

$$A = \frac{CY_0 r^2}{Q(Y_0 J_0' - J_0 Y_0')}\, dr + a \qquad \ldots (13)$$

$$B = \frac{CJ_0 r^2}{Q(Y_0 J_0' - J_0 Y_0')}\, dr + b \qquad \ldots (14)$$

As limit $r \longrightarrow 0$, $\bar{u}\ (r,s)$ should be finite, and at $r = \beta$ no slip condition should be satisfied. This leads to:

$$\bar{u} = D_1 J_0(Qr)\, Y_0(Qr) - u_0 \qquad \ldots (15)$$

where

$$D_1 = \frac{C}{Q^3}\left(r^2 - \frac{2}{Q^2}\right) + \frac{2C}{Q^5}\log\left(r^2 - \frac{2}{Q^2}\right) + a$$

Now the value of u (as limit $r \longrightarrow o, u(r,s)$ should be finite and at $r \longrightarrow \beta$ no slip condition should be satisfied), can be evaluated using Laplace inversion formula, as follows:

$$u(r,t) = \frac{1}{2\pi i}\int_{r-\infty}^{r+\infty} \frac{1}{(s+)}\left(D_1 J_0(Qr) - Y_0(Q_r) - \bar{u}_0\right) e^{st}\, ds \qquad \ldots (16)$$

Therefore,

$$\begin{aligned} u(r,t) = \frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} \Big[& \frac{s^{5/2}}{(s^2+Bs-BK_2)^{5/2}}\left(r^2(s^2+Bs-BK_2)-2s\right) \\ & +2\log\frac{(r^2(s^2+Bs-BK_2)-2s)}{(s^2+Bs-BK_2)} + \frac{\bar{u}_0}{s} - \left(1+\frac{1}{2}\log(B,\beta^2)\right) \\ & +\frac{1}{2}\left(\frac{s}{B}-\frac{K_2}{s}\right) - \frac{1}{4}\left(\frac{s}{B}-\frac{K_2}{s}\right)^2 \left(\frac{1}{s+}\right) \\ & \Big\{1-\frac{1}{2}\log(B.r^2) - \frac{1}{2}\left(\frac{s}{B}-\frac{K_2}{s}\right) + \frac{1}{4}\left(\frac{s}{B}-\frac{K_2}{s}\right)^2 \Big\}\Big] e^{st}\, ds \end{aligned}$$

Then u(r,t) has been solved numerically by using inversion formula of Laplace transformation with the help of computer program [6]. Similarly v(r,t) can be found from equation (9) along with u(r,t). The skin friction on unit area of the surface of cylinder due to fluid is given by

$$F = \mu \left[\frac{\partial u}{\partial r} \right] r = 1 \qquad \ldots (17)$$

Once the value of v is obtained, the skin friction is also found out by equation (17), by putting the value of v.

Velocities of liquid and solid particles and the skin friction on the wall have been represented grahically with variable axial positions, time and number densities of the suspension.

From figure (1) it is clear that the velocity of the fluid decreases as the number density of the dust particles increases. Velocity of the fluid also decreases with the axial position from the throat and the time that elapses. Fluid velocity is always maximum on the central axix.

From figure (2) it is clear that the particle velocity decreases as the number density increases. Particle velocity also decreases with axial position from the throat and the time elapsed. Further, from figure (1) and (2), we infer that at a particular axial position the fluid velocity is greater than the particle velocity. Near the boundary, the dust particles exhibit sudden changing velocity and this tendency decreases with the number density.

From figure (3) we find that skin friction increases with number density and decrease with time and axial position from the throat. Thus, in order to have lower values of the skin friction for blood type suspension the number density should be decreased.

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This is frequently done by the dilution process in medical sciences. The above inferences from the mathematical model are characterized in a much more comprehensive manner by the statistical analyses in the following section.

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4. STATISTICAL ANALYSES :

For the data generated per the mathematical model as displayed in figures 1, 2 and 3, simple statistical analyses were carried out. Multiple regression of u, v and F each on the other relevent independent variables and various partial correlation coefficient ([4],[5], & [7]) were computed in order to bring out the dependence structures

First we take to 'u', i.e., the velocity of the fluid.

When $N = 4 \times 10^7$ ·

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(a) Case 1 : z = 0 : It is found that,

$$u = 5.113 - 0.129t - 5.208r. \quad \text{... (18)}$$

Also, the partial correlations,

$$R_{rt.u} = -0.051 \; ; \; R_{ut.r} = -0.994 \; ; \; R_{ur.t} = -0.052 \quad \text{...(19)}$$

(b) Case 2 : z = 2 : we find that,

$$u = 2.044 - 0.252t - 2.084r. \quad \text{... (20)}$$

And,

$$R_{rt.u} = -0.216 \; ; \; R_{ut.r} = -0.992 \; ; \; R_{ur.t} = -0.218 \quad \text{... (21)}$$

When, $N = 5 \times 10^7$:

(a) Case 1 : $z = 0$: We find that,

$$u = 5.861 - 0.752t - 5.237r. \quad \dots (22)$$

And $\quad R_{rt.u} = -0.126$; $R_{ut.r} = -0.968$; $R_{ur.t} = -0.131 \quad \dots (23)$

(b) Case 2 : $z = 2$: We find that,

$$u = 2.357 - 0.088t - 2.277r. \quad \dots (24)$$

And

$R_{rt.u} = -0.088$; $R_{ut.r} = -0.995$; $R_{ur.t} = -0.088 \quad \dots (25)$

So (18), (20), (23) and (25) each show the average mathematical way 'u' goes down with 't' and /or 'r' going up. Also, as per (19), (21), (24), &(25)'u' and 't' are significantly related (opposite way) while the other relations are insignificant.

Now we take to 'v', i.e., velocity of solid particles.

When, $N = 4 \times 10^7$:

(a) Case 1 : $z = 0$ we find that

$$v = 0.769 - 0.121t - 0.679r \quad \dots (26)$$

And

$R_{rt.v} = -0.114$; $R_{vt.r} = -0.944$; $R_{vr.t} = -0.121 \quad \dots (27)$

(b) Case 2 : $z = 2$: We find that,

$$v = 1.676 - 0.276t - 2.147r.$$

And,

$R_{rt.v} = -0.022$; $R_{vt.r} = -0.711$; $R_{vrr.t} = -0.311 \quad \dots (28)$

When, $N = 5 \times 10^7$:

(a) Case 1 : $z = 0$: We find that,

$$v = 0.845 - 0.132t - 0.755r. \quad \ldots (29)$$

And,

$$R_{rt.v} = 0.143 \; ; \; R_{vt.r} = -0.963 \; ; \; R_{vr.t} = -0.149 \quad \ldots (30)$$

(b) Case 2 : $z = 2$: We find that,

$$v = 0.346 - 0.110t - 0.338r.$$

And,

$$R_{rt.v} = -0.643 \; ; \; R_{vt.r} = -0.996 \; ; \; R_{vr.t} = -0.646 \quad \ldots (31)$$

The structures of the relationship are quite similar to those 'u' except that the correlations between 'r' and 't' and 'r' and 'u' is relatively not that insignificant as in case of 'u'.

Now we take to 'F', i.e. the skin friction.

When, $N = 4 \times 10^7$: We find that,

$$F = 0.480 - 0.087t - 0.163z. \quad \ldots (32)$$

and,

$$R_{zt.F} = -0.370 \; ; \; R_{Ft.z} = -0.995 \; ; \; R_{Fz.t} = -0.372 \quad \ldots (33)$$

When, $N = 5 \times 10^7$: We found that,

$$F = 0.688 - 0.095t - 0.205z. \quad \ldots (34)$$

And,

$$R_{zt.F} = -0.187 \; ; \; R_{Ft.z} = -0.985 \; ; \; R_{Fz.t} = -0.189 \quad \ldots (35)$$

Here also like for the case of 'u' only 'F' and 't' are significantly related the other pairs, namely, 'z' and 't' and 'F' and 'z' are not so significantly related.

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LATTICE SUM OF ELECTRIC FIELD GRADIENTS IN ORTHORHOMBIC CRYSTAL

B.Kumar*, H.C. Verma* and D.P. Verma**

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ABSTRACT

Electric field gradient (FEG) is calculated at a nuclear site in the Simple Orthorhombic crystal. The method uses Euler Maclaurin (EM) Summation formula and makes Plane Wise summation in direct crystal space.

Key words : Electric field gradient, Lattice Sum Convergence, Simple Orthorhombic Crystal, Euler-Maclaurin formula, Double Series.

1. INTRODUCTION

In a previous paper (Verma et al. 1983) a new technique to sum the electric field gradient (EFG) over lattice points at a nuclear site in simple tetragonal crystal was introduced. The technique involves repeated use of Euler-Maclaurin (EM) formula to evaluate the infinite series and the same formula was used to estimate error terms from a related finite sum. The lattice sum is evaluated inthe direct crystal space keeping the numerical computations reasonably low, unlike most of the previous methods using Fourier transform in reciprocal space.

In the present paper we use the above method (Verma et al. 1983) to calculate the Electric Field Gradient (EFG) at the nuclear site in simple orthorhombic crystal.

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2. DETAILS OF CALCULATION

The lattice part of electric field gradient (EFG) eq_{latt} is defined as

$$eq_{latt} = \sum_{\lambda} \frac{ze}{4\pi e_o} \frac{3z_{\lambda}^2 - r_{\lambda}^2}{r_{\lambda}^5} - \frac{ze}{4\pi e_o V} \int \frac{3z^2 - r^2}{r^5} d^3r \quad \ldots(1)$$

where,

z = charge on each ion

e = electronic charge

e_o = permittivity of vacuum

z = z-co-ordinate of the ion at th lattice point

r = distance of lattice point from the origin situated at the nucleus of interest

V = volume of the unit cell

The integration refers to the contribution from an assumed uniform background of negative charges. Both the terms in equation (1) are conditionally convergent, hence same boundary shape at infinity is to be assumed in summation and integration.

We employ plane wise summation which leads to a rapidly convergent expression of the sum. This means the crystal is assumed to be slab shaped with faces perpendicular to z-direction. With this boundry shape the integration $\int \frac{3z^2 - r^2}{r^5} d^3r$ in equation (1) for the assumed boundary shape comes out to be $(-8\pi/3)$ (de Wette 1961).

In an orthorhombic lattice the lattice points are written as (n_1a, n_2b, n_3c), where n_1, n_2, n_3 run through all integers.

The summation in equation (1) can be written as

$$S = \frac{ze}{4\pi e_o a^3} \sum_{n_1, n_2, n_3 = -\infty}^{\infty} f(n_1, n_2 n_3) \quad \ldots (2)$$

where,

$$f(n_1,n_2,n_3) = \frac{2n_3B^2 - n_1^2 - n_2^2 A^2}{(n_1^2 + n_2^2A^2 + n_3^2 B^2)^{5/2}},$$

$A = b/a$ and $B = c/a$

The summation over n_1 and n_2 are to be carried out before n_3 summation.

2.1 Contribution from base plane (z = $n_3c = 0$)

The Contribution to [$\frac{4\pi e_o a^3}{ze}$ S] from the base plane is

$$U_o = \sum_{n_1,n_2=-\infty}^{\infty} g(n_1,n_2)$$

$$= -4 \sum_{n_1=1}^{\infty} \sum_{n_2=0}^{\infty} g(n_1,n_2) + 2 \sum_{n_1=1} g(n_1,0)$$

$$- \frac{2}{A^3} \sum_{n_2=1}^{\infty} g(0,n_2) \qquad \ldots (3)$$

where,

$$g(n_1,n_2) = \frac{1}{(n_1^2+n_2^2A^2)^{3/2}}$$

The Euler-Maclaurin (EM) Summation formula (Hildebrand 1956) is

$$\sum_{n=M} f(n) = \int_M^N f(x)dx + \frac{1}{2} \{f(N) + f(M) + \int_M^N P_1(x) \frac{df(x)}{dx}dx \qquad \ldots(4)$$

where $P_1(x) = x - 1/2$ for $0 < X \leq 1$ and is defined outside this region by $P_1(x+1) = P_1(x)$.

Applying the EM formula (4) to (3) we get,

$$U_o = \frac{4}{A} \sum_{n_1=1}^{\infty} \frac{1}{n_1^2} \quad \frac{2}{A^3} \sum_{n_1=1}^{\infty} \frac{1}{n_1^3} - 4 \int_0^{\infty} P_1(x) \sum_{n_1=1}^{\infty} R\, dx$$

in which $R = \frac{d}{dx} g(n_1, x)$

Thus $$U_o = -\frac{4}{A}\,\xi(2) - \frac{2}{A^3}\,\xi(3) + \epsilon_o(\infty) \qquad \text{... (5)}$$

Where $\epsilon_o(\infty)$ contains all integrals involving $P_1(x)$ terms and $\xi(2) = \pi^2/6$, $\xi(3) = 1.20205690$ are the Riemann Zeta functions at 2 and 3 respectively.

To Estimate $\epsilon_o(\infty)$ we apply the above procedure to the finite sum

$$U_o(N) = -\sum_{n_1, n_2=-N}^{N} g(n_1, n_2)$$

$$= -4\sum_{n_1=1}^{N}\ \sum_{n_2=0}^{N} g(n_1, n_2) + 2\sum_{n_1=1}^{N} g(n_1, 0)$$

$$-\frac{2}{A^3}\sum_{n_2=1}^{N} g(0, n_2)$$

and get,

$$U_o(N) = -4\sum_{n_1=1}^{N} G(n_1, N) - \frac{2}{A^3}\sum_{n_2=1}^{N} g(0, n_2)$$

$$-2\sum_{n_1=1}^{N} g(n_1, N) - 4\int_0^N P_1(x)\sum_{n_1=1}^{N} R\,dx$$

Hence,

$$\epsilon_o(N) = -4\int_0^N P_1(x)\sum_{n_1=1}^{N} R\,dx$$

$$= -4\sum_{n_1=1}^{N}\ \sum_{n_2=0}^{N} g(n_1, n_2) + 4\sum_{n_1=1}^{N} G(n_1, N)$$

$$+ 2\sum_{n_1=1}^{N} g(n_1, 0) + 2\sum_{n_1=1}^{N} g(n_1, N) \qquad \text{.......(6)}$$

where,

$$G(n_1\ ,\ N)\ =\ \frac{N}{n_1{}^2\ (n_1{}^2\ +\ N^2\ A^2\ )^{1/2}}$$

As N increases | $\epsilon_o\ (\infty) - \epsilon_o\ (N)$ | decreases. It was shown by Verma et al. (1993) that the absolute difference between ( ) and (N) for a similar expression reduces below 10 as N is taken greater than 80. Our expression has a similar nature and hence we expect that to the accuracy of present calculations (five significant digits) | $\epsilon_o\ (\infty) - \epsilon_o\ (80)$ | may be neglected. Hence we calculate (N) for N = 80 from equation (6) and substitute it for (oo) in equation (5) to get the contribution due to base plane.

2.2 Contribution from the plane $z = n_3c\ (n \neq 0)$

The contribution of the plane $Z=n_3c$ to $(4\pi\epsilon a_o^3 S/Ze)$ is

$$U_k = \sum_{n_1,n_2=-\infty}^{\infty} f(n_1,n_2,n_3)$$

$$= 4\sum_{n_1=1}^{\infty}\ \sum_{n_2=0}^{\infty} f(n_1,n_2,n_3) - 2\sum_{n_1=1}^{\infty} f(n_1,0,n_3)$$

$$+ 2\sum_{n_2=0}^{\infty} f(0,n_2,n_3) - \frac{2}{K^3} \qquad \ldots (7)$$

where $K = n_3B$

Using EM formula (4) on summation over n_2 we get,

$$U_k = \frac{4}{A}\sum_{n_1=1}^{\infty}\frac{2K^2}{(K^2+n_1^2)^2} - \frac{4}{A}\sum_{n_1=1}^{\infty}\frac{1}{(K^2+n_1^2)} + \frac{2}{AK^2}$$

$$+ 2\int_0^{\infty} P_1(x)R_2dx + 4\int_0^{\infty} P_1(x)\sum_{n_1=1}^{\infty} R_1dx$$

$$= \frac{4}{A}\sum_{n_1=0}^{\infty}\frac{2K^2}{(K^2+n_1^2)^2} - \frac{4}{A}\sum_{n_1=0}^{\infty}\frac{1}{(K^2+n_1^2)} - \frac{2}{AK^2}$$

$$+ 2\int_0^{\infty} P_1(x)\, R_2 dx + 4\int_0^{\infty} P_1(x) \sum_{n_1=1}^{\infty} R_1 dx \qquad \ldots (8)$$

where $R_1 = \frac{d}{dx} f(n_1,x,n_3)$ and $R_2 = \frac{d}{dx} f(0,x,n_3)$

The first two series of equation (8) can be summed exactly using Cauchy's Residue formula (Spiegel 1964).

$$\sum_{n=0}^{\infty} \frac{1}{(K^2+n^2)^2} = \frac{\pi}{4}\left[\frac{\coth(\pi K)}{K^3} \quad \frac{\operatorname{cosech}^2(\pi K)}{K^2}\right] + \frac{1}{2K^4}$$

and

$$\sum_{n=0}^{\infty} \frac{1}{K^2+n^2} = \frac{1+\pi K \coth(\pi K)}{2K^2}$$

So that equation (8) becomes

$$U_K = \frac{2\pi^2}{A} \operatorname{cosech}^2(\pi K) + e_k(\infty) \qquad \ldots (9)$$

where $e_k(\infty)$ contains all integrals involving $P_1(x)$ terms.

Again to estimate $e_k(\infty)$ we repeat the above procedure with the finite sum

$$U_k(N) = \sum_{n_1,n_2=-N}^{N} f(n_1,n_2,n_3)$$

$$= 4\sum_{n_1=1}^{N}\sum_{n_2=0}^{N} f(n_1,n_2,n_3) - 2\sum_{n_1=1}^{N} f(n_1,0,n_3)$$

$$+ \sum_{n_2=0}^{N} f(0,n_2,n_3) - \frac{2}{K^3}$$

and get

$$e_K(N) = 2\int_0^N P_1(x)\, R_2 dx + 4\int_0^N P(x) \sum_{n_1=1}^{N} R_1 dx$$

$$= \sum_{n_1,n_2=-N}^{N} f(n_1,n_2,n_3) + H(n_1,n_3,N) \qquad \ldots (10)$$

where,

$$H(n_1,n_3,N) = -\sum_{n_1=1}^{N} \frac{12\,K^2N}{(K^2+n_1^2)^2(K^2+n_1^2+N^2A^2)^{1/2}}$$

$$+\sum_{n_1=1}^{N} \frac{4K^2\,N^3\,A^2}{(K^2+n_1^2)\,(K^2+n_1^2+N^2A^2)^{3/2}}$$

$$+\sum_{n_1=1}^{N} \frac{4\,N}{(K^2+n_1^2)\,(K^2+n_1^2A^2)^{\frac{1}{2}}}$$

$$-2\sum_{n_1=1}^{N} f(n_1,n_3,N) - f(0,n_3.N)$$

$$-\frac{4\,N}{K^2(K^2+N^2A^2)^{\frac{1}{2}}}\;\frac{2\,N^3\,A^2}{K^2(K^2+N^2A^2)^{3/2}}$$

We have set N=80, and evaluated $\epsilon_k(N)$ from equation (10). This is substituted for $\epsilon_k(\infty)$ in equation (9) thereby neglecting $|\,\epsilon(\infty)-\epsilon_k(N)\,|$ and obtaining U_k.

3. Results and discussion

We made calculations for eight orthorhombic structures with different ratios of a:b:c (listed in table-1). Sets of lattice parameters correspond to the lattices of orthorhombic elements B, Ga, P(black), O, S(yellow), Br, I and U (Darken Lawrence et al.1953).

The contributions from base plane and other planes were evaluated from the equations derived above. The contribution falls rapidly as n_3 increases and that is the speciality of this method. In all our claculations a maximum of six planes above the origin need to be considered. The contribution from the planes below the origin was equal to that from the planes above by symmetry. The contribution $(-8\pi/3 \ \frac{ze}{4\pi e_o a^3})$ was added to get the eq_{latt} in c-direction.

For each set of lattice parameters the calculations were repeated thrice by choosing the z axis along each of the three edges of the cell and hence to get the EFG's along the three direction. These directions were then labelled according to the convention $|V_{zz}| \geq |V_{yy}| \geq |V_{xx}|$. The sum $V_{xx} + V_{yy} + V_{zz}$ is found to be zero within the limits of accuracy of present calculations as is expected from the Laplace equation.

As the total computations involved is of low order and the sum is made in direct crystal space without any special regrouping of charges, the use of Euler Maclaurin summation formula difinitely has interesting application in lattice summation exercises.

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TABLE - 1

Lattice EFG in Orthorhombic Lattice Structures at Substitutional Sites in units of Ze/4 C_o

Element	Lattice Parameters of the simple orthorhombic Lattice (in K_x units)			EFG		
	a	b	c	V_{xx}	V_{yy}	V_{zz} = eq
B	8.93	10.13	17.86	0.0013986	0.0039938	0.0053920
Ga	4.5107	4.516	7.6448	0.022063	0.022318	-0.044399
P (black)	3.31	4.38	10.50	0.010841	0.10415	-0.11499
O	3.44	3.82	5.50	0.017592	0.056615	-0.074201
S (yellow)	10.48	12.92	24.55	0.00047180	0.0028624	-0.0033361
Br	6.67	8.72	4.48	-0.0098774	-0.027391	0.037269
I	7.2501	9.7711	4.7761	-0.0082761	-0.023389	0.031667
U	5.865	4.945	2.852	-0.061142	-0.095374	0.15652

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EULER MACLAURIN SUMMATION FORMULA FOR TWO DIMENSIONS

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ABSTRACT

We here develop Euler Maclaurin Summation Formula (EMSF) for two dimension case so that it can be used in summing over a set of lattice points in a plane. Other three versions of the formula are also got.

AMS(MOS) Subject classifications (1980): 65 B 15

Key words : Lattice sum, Discrete mathematics, Double series summation.

1. INTRODUCTION

The EMSF for one dimension [1] is

$$\sum_{r=1}^{n} f(r) = \int_{1}^{n} f(x)dx + \frac{1}{2}\{f(1) + f(n)\} + \int_{1}^{n} P_1(x)\, f'(x)\, dx$$

where $P_1(x)$ is Bernoulli's polynomial of order one. But no such result like Euler Maclaurin Summation Formula (EMSF) is available in problems of two and three dimensions. We here develop the formula for two dimensions and similar extension for three dimensions is easy.

2. FORMULATION OF THE PROBLEM

Here we find the value of

$$S = \sum_{r=1}^{m} \sum_{s=1}^{n} f(r,s)$$

we first fix r and let s vary from 1 to n, thus

$$S = \sum_{r=1}^{m} \int_{1}^{n} f(r,y)dy + \frac{1}{2} \sum_{r=1}^{m} f(r,1) + \frac{1}{2} \sum_{r=1}^{m} f(r,n)$$

$$+ \sum_{r=1}^{m} \int_{1}^{n} P_1(y) f_y(r,y)dy$$

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we apply EMSF to each term of the R.H.S. and after arranging, get

$$S = \int_1^m \int_1^n f(x,y,)\,dxdy + \frac{1}{4}\{f(1,1)+f(m,1)+f(1,n)+f(m,n)\}$$

$$+\frac{1}{2}\{\int_1^n f(1,y)dy + \int_1^n f(m,y)dy + \int_1^m + f(x,1)dx + \int_1^m f(x,n)dx\}$$

$$+\frac{1}{2}[\int_1^n P_1(y)f_y(1,y)dy + \int_1^m P_1(x)f_x(x,1)dx +$$

$$+\int_1^n P_1(y)f_y(m,y)dy + \int_1^m P_1(x)f_x(x,1)dx]$$

$$+\int_1^m \int_1^n \{P_1(x)f_x(x,y) + P_1(y)f_y(x,y)\}dxdy$$

$$+\int_1^m \int_1^n PP_1(x)P_1(y)f_{xy}(x,y)dxdy \qquad \ldots (1)$$

provided the partial derivatives used exist and are continuous.

In Green's theorem

$$\iint_D (Q_x - P_y)dxdy = \int_C (P\,dx + Q\,dy)$$

let C be the rectangle with vertices (1,1), (m,1), (m,n), (1,n) let D be the part of the plane enclosed by C and let $Q = -f(x,y) = -P$. then

$$\iint_D \{f_x(x,y) + f_y(x,y)\}\,dxdy = \int_C f(x,y)\ (dy-dx)$$

$$= -\int_1^m f(x,1)dx + \int_1^n f(m,y)dy + \int_1^m f(x,n)dx - \int_1^n f(1,y)dy \qquad \ldots (2)$$

Next we put $P = -P_1(x)f_x(x,y)$ and $Q = P_1(y)f_y(x,y)$, then

$$\int_1^m \int_1^n \{P_1(y)f_{xy}(x,y)+P_1(x)f_{xy}(x,y)\}dxdy$$

$$= -\int_1^m P_1(x)f_x(x,1)dx + \int_1^n P_1(y)f_y(m,y)dy + \int_1^m P_1(x)f_x(x,n)dx$$

$$- \int_1^n P_1(y)f_y\,(1,y)dy. \qquad \ldots (3)$$

Thus (1) becomes, after putting values from (2) and (3)

$$S = \int_1^m \int_1^n \{f(x,y) - \frac{1}{2}\, f_x(x,y) - \frac{1}{2}\, f_y(x,y)\}dxdy$$

$$+ \frac{1}{4}\{f\,(1,1) + f\,(m,1) + f(1,n) + f\,(m,n)\}$$

$$+ \int_1^m \{f\,(x,n)\; dx + \int_1^n f(m,y)\; dy$$

$$+ \int_1^m P_1(x)\; f_x(x,n)\; dx + \int_1^n P_1(y)f_y(m,y)\; dy$$

$$+ \int_1^m \int_1^n \{P_1(x)f_x\;(x,y) + P_1(y)f_y\;(x,y)\}\; dxdy$$

$$+ \int_1^m \int_1^n [P_1(x)\; P_1\,(y) - \frac{1}{2}\; P_1(x) - \frac{1}{2}P_1(y)]f_{xy}(x,y)dx\,dy$$

$$\ldots (4)$$

This is our formula in most general case. We now try to discuss EMSF in two dimension for skew-symmetric and symmetric functions. We know that every function can be expressed as a sum of a symmetric and skew-symmetric function. Hence let

$$f(x,y) = g\,(x,y) + h(x,y),$$

g(x,y) being symmetric and h(x,y) skew symmetric.

The terms or

$$\sum_{r,s=1}^{n} h(r,s)$$

can be grouped in pairs of type h(r,s) + h(s,r) and we are left with terms of the type h(r,r), each of which is zero by skew symmetric property. Thus

$$\sum_{r,s=1}^{n} f(r,s) = \sum_{r,s=1}^{n} g(r,s)$$

Now g is symmetric i.e. g(x,y) = g(y,x) Differentiating, $g_x(x,y) = g_x\,(y,x)$. Using these results and putting m=n , (1) reduces to

$$\sum_{r,s=1}^{n} f(r,s) = \sum_{r,s=1}^{n} g(r,s)$$

$$= \int_1^n\int_1^n g(x,y)\,dxdy + \frac{1}{4}\,\{g(1,1) + 2g(n,1)$$

$$+ g(n,n)\} + \{\int_1^n g(x,1)\,dx + \int_1^n g(x,n)\,dx\}$$

$$+ \{\int_1^n P_1(x)\, g_x(x,1)\,dx$$

$$+ \int_1^n P_1(x)\, g_x(x,n)\,dx\}$$

$$+ 2\int_1^n\int_1^n P_1(x)\, g_x(x,y)\,dxdy$$

$$+ \int_1^n\int_1^n P_1(x)\,P_1(y)\, g_{xy}(x,y)\,dxdy \qquad (5)$$

The last result can be written in terms of 3rd order Bernoulli polynomials if we integrate by parts twice the terms containing $P_1(x)$, and note that $P_3(x)$ is zero at integral arguments. So

$$\int_1^n P_1(x)\,\{g_x(x,1) + g_x(x,n)\}\,dx$$

$$= \frac{B_2}{2!}\{g_x(n,n) - g_x(1,1)\} + \int_1^n P_3(x)\,[g_{xxx}(x,1)$$

$$+ g_{xxx}(x,n)]\,dx \qquad \ldots (6)$$

$$\int_1^n\int_1^n P_1(x)\, g_x(x,y)\,dxdy$$

$$= \frac{B_2}{2!}\int_1^n \{g_x(n,y) - g_x(1,y)\}\,dy$$

$$+ \int_1^n \int_1^n P_3(x) g_{xxx}(x,y)\, dxdy \qquad \ldots (7)$$

$$\int_1^n \int_1^n P_1(x)\, P_1(y)\, g_{xy}(x,y)\, dxdy$$

$$= \frac{B_2}{2\,!} \{g_{xy}(n,n) - 2g_{xy}(n,1) + g_{xy}(1,1)\}$$

$$+ B_2 \int_1^n P_3(x)\, \{g_{xxxy}(x,n) - g_{xxxy}(x,1)\}\, dx$$

$$+ \int_1^n \int_1^n P_3(x)\, P_3(y)\, g_{xxxyyy}(x,y)\, dxdy \qquad \ldots (8)$$

provided the partial derivatives used exist and are continuous so that change of order of differentiation is permitted.

Putting in (5) the above values as obtained in (6), (7) and (8) and arranging, we have,

$$\sum_{r,s=1}^{n} g(r,s)$$

$$= \int_1^n \int_1^n g(x,y)dxdy + \frac{1}{4}\{g(1,1) + 2g(n,1) + g(n,n)\}$$

$$+ [\ \frac{B_2}{2!}\{g_x(n,n) - g_x(1,1)\}$$

$$(\frac{B_2}{2!})^2 \ \{g_{xy}(n,n) - 2g_{xy}(n,1) + g_{xy}(1.1)\}]$$

$$+ \int_1^n \{g(x,1) + g(x,n) \ \ dx$$

$$+ B_2 \int_1^n \{g_x(n,y) - g_x(1,y)\}\, dy$$

$$+ \int_1^n P_3(x)\{g_{xxx}(x,1) + g_{xxx}(x,n)\}\, dx$$

$$+ B_2 \int_1^n P_3(x)\ \{g_{xxxy}(x,n) - g_{xxxy}(x,1)\}\, dx$$

$$+ 2 \int_1^n \int_1^n P_3(x) g_{xxx}(x,y)dxdy$$

$$+ \int_1^n \int_1^n P_3(x)P_3(y)g_{xxxyyy}(x,y)dxdy \qquad \ldots (9)$$

3. APPLICATION

A number of physical quantities are expressed as a negative powers of distance. Matter is a collection of discrete atoms. In rough calculation, matter is taken to be continuous and the physical quantities are calculated using double or triple integrals. This is done because much dificult mathematics is required if we consider the atoms at the positions where they are actually situated. Some people attack such a situation by a number transformations. If one wants to avoid transformations and avoid the assumption of continuous matter then one way of handling the situation is through the theorem developed in this paper. Here we give one such application using fourth power of distance.

I. J. Zukker and M.M. Robertson [2] calculated

$$\sum_{(m,n)\neq(0,0)}^{\infty} \frac{1}{(m^2+n^2)^2} = 6.0267002 \pm .00207895$$

using a number of Transformations and L functions. We, in order to calculate the above sum, take

$$\sum_{(m,n)\neq(0,0)}^{\infty} \frac{1}{(m^2+n^2)^2} = 4\zeta(4)+4 \sum_{m,n=1}^{\infty} \frac{1}{(m^2+n^2)^2}$$

$$= 4\zeta(4) + 4 \sum_{m,n=1}^{5} \frac{1}{(m^2+n^2)^2}$$

$$+ 8 \sum_{n=1}^{5} \sum_{m=6}^{\infty} \frac{1}{(m^2+n^2)^2} + 4 \sum_{m,n=6}^{\infty} \frac{1}{(m^2+n^2)^2}$$

The second term is a finite sum, the third has been calculated using EMSF for one dimension and the last one using our theorem for two dimensions. The result got is

$$6.0265284 \pm 0.005736$$

which compares well with [2].

4. REMARKS

The formula has a wide potentiality of application as most physical quantities are expressible as a point function is two or three dimensions. In many cases, form (2) is more suitable for numerical calculations than form (1).

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$$\sum_{(m,n)\neq(0,0)}^{\infty} (am^2 + bnm + cn^2)^{-s}$$

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ON SOME SELF-SUPERPOSABLE FLUID MOTIONS IN PROLATE SPHEROIDAL DUCTS

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ABSTRACT

In the present paper an attempt has been made to find some self-superposable motions of incompressible fluid in prolate spheroidal ducts. No stress has been given on boundary conditions and the solutions thus determined contain a set of constants. Pressure distribution of some of such flows has also been discussed. The curves along which the vorticity of the flow becomes constant have also been attempted. Some irrotational flows and the surfaces along which the flows show the tendency of irrotationality has also been determined. The aim of the paper is to introduce a method for solving the basic equations of fluid dynamics in prolate spheroidal coordinates by using the property of self-superposability.

1. INTRODUCTION

It was shown by Ballabh [2] that when a flow with velocity q satisfies the condition

$$\text{Curl } (\bar{q} \times \text{curl } \bar{q}) = 0 \quad \ldots (1)$$

it becomes self-superposable. In the present paper the authors have attempted some fluid velocities for which $(\bar{q} \times \text{Curl } \bar{q}) = \bar{p}$ (say) can be represented as the gradient of a scalar quantity. Attention has been focussed on incompressible fluids only. For such fluids some velocities have been determined which satisfy the condition in prolate spheroidal system of coordinates and

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for which $\overline{p}$ can be represented as the gradient of a scalar quantty θ (say). The solutions thus found must be of self-superposable type. Each will have a set of constants which can be determined by the boundary conditions.

It has further been shown by Ballabh (3) that if $\overline{q}_1$ and $\overline{q}_2$ are self-superposable flows and $\overline{q}_1 \pm \overline{q}_2$ are also self-superposalbe then $\overline{q}_1$ and $\overline{q}_2$ are mutually superposable. By using this property some more self-superposable flows have been determined. Attempts have also been made to find some curves along which the vorticity of the flow becomes constant and also the conditions of irrotationality.

In this paper a method will be introduced to solve the basic equations of fluid dynamics in prolate spheroidal coordinates by using the property of self superposability. Mittal [5,6] Mittal et al.[7,8,9,10] have recently solved the equations for different coordinate systems.

FORMULATION OF THE PROBLEM

Let $\overline{q} \times \text{Curl } \overline{q} = \overline{p}$ ...(2)

For incompressible fluids, we have

$$\text{div } \overline{q} = 0 \qquad \text{... (3)}$$

Now if a solution of equation (3) be found in such a way that p can be represented as the gradient of a scalar quantity say θ, i.e.,

$$\overline{p} \ \text{grad } \theta, \qquad \text{... (4)}$$

it will give a self-superposable flow. It has been shown by Agarwal [1] that such solutions will also satisfy the equation of motion for a steady flow. For determining a flow of liquid let us consider the flow in prolate spheroidal coordinates (u, v, w). If q_u, q_v, q_w be the components of q at any point (u, v, w) in prolate spheroidal coordinates [11] then in order to make equation (3) integrable we may consider the following cases :

Case I

Let $q_u = 0$. In this case equation (3) will be satisfied by a solution.

$$\begin{aligned} q_u &= 0 \\ q_v &= \frac{AU(u)\, W(w)}{\sinh u.\ \sin v \sqrt{(\sinh^2 u + \sin^2 v)}} \qquad \ldots (15) \\ q_w &= \frac{BU_1(u)\, V(v)}{(\sinh^2 u + \sin^2 v)} \end{aligned}$$

where $U(u)$, $U_1(u)$ are integrable functions of u, $V(v)$ and $W(w)$ the integrable functions of v and w respectively, and A, B the constants.

For this fluid velocity, it can be shown that $\bar{p}$ can be represented by the gradient of a scalar quantity θ given by

$$\theta = \frac{A^2 W^2}{a \sin^2 v} \int \frac{U\, U'\, du}{\sinh^2 u\, (\sinh^2 u + \sin^2 v)^{3/2}}$$

$$- \frac{A^2\, W^2}{a \sin^2 v} \int \frac{U^2 \operatorname{cosech} u.\ \operatorname{Coth} u}{(\sinh^2 u + \sin^2 v)^{3/2}}\, du$$

$$+ \frac{B^2 U_1^2}{a \sinh u} \int \frac{V\, V' \quad dv}{\sin v\, (\sinh^2 u + \sin^2 v)^2}$$

$$+ \frac{B^2\, U_1^2}{a \sinh u} \int \frac{V^2\, (\sinh^2 u - \sin^2 v) \cos v}{(\sinh^2 u + \sin^2 v)^3 \sin^2 v}\, dv$$

$$+ \frac{A\,B\,U\,U_1}{a(\sinh^2 u+\sin^2 v)^4} \quad [\{V' + V \operatorname{Cot} v\} (\sinh^2 u+\sin^2 v) + V \sin 2v] \int W\, dw$$

$$- \frac{A^2\, U^2}{a(\sinh^2 u+\sin^2 v)^2 \sinh^2 u \sin^2 v} \int W\, W'\, dw \quad . \qquad \ldots (6)$$

here U(u), U_1(u), V(v), W(w) are represented by U, U_1, V and W respectively and U', U'_1, V' and W' represent their differentials.

By choosing different suitable sets of values of U, U_1, V and W we may get a number of self-superposable fluid velocities. One of such velocities can be obtained by taking

$$U_1 = U = a \sinh u,\ V = a \sin v,\ W = a \cos w$$

$$\text{and } B = A \qquad \ldots\ldots (7)$$

the fluid velocity will become ;

$$q_u = 0$$

$$q_v = \frac{A \operatorname{Cos} w}{\sin v \sqrt{(\sinh^2 u+\sin^2 v)}} \qquad \ldots\ldots (8)$$

$$q_w = \frac{A \sinh u .\ \sin v}{(\sinh^2 u + \sin^2 v)}$$

and ;

$$\theta = \frac{A^2}{a} \left[\frac{\operatorname{Cos}^2 w}{\sin^4 v} \left\{ \frac{1}{2 \sin v} \log \left(\frac{\sqrt{\sinh^2 + \sin^2 v} - \sin v}{\sqrt{\sinh^2 u+\sin^2 v} + \sin v} \right) - \right.\right.$$

$$\left. \frac{1+\sinh u}{\sqrt{\sinh^2 u + \sin^2 v}} \right\} + 2 \sinh u \left\{ \frac{\sqrt{\sinh^2 u+\sin^2 v} + \sinh^2 u}{\sqrt{\sinh^2 u + \sin^2 v}} \right\}^{1/2}$$

$$+ \frac{\sinh^2 u \cos v \sin w (2 \sinh^2 u + 3 \sin^2 v)}{(\sinh^2 u + \sin^2 v)}$$

$$- \frac{\sin^2 w}{2(\sinh^2 u + \sin^2 v)^2} \qquad \ldots (9)$$

If U, U_1, V, W are constants then

$$q_u = 0$$

$$q_v = \frac{C_1}{\sinh u . \sin v \sqrt{\sinh^2 u + \sin^2 v}} \qquad \ldots (10)$$

$$q_w = \frac{D_1}{(\sinh^2 u + \sin^2 v)}$$

and

$$\theta = \frac{D_1^2}{a \sin^6 v} \left[\frac{3 \sin^{-1}}{2 \sin v} \left(\frac{\sin v}{\sqrt{\sin^2 u + \sin^2 v}} \right) - \frac{\sinh u}{2(\sinh^2 u + \sin^2 v)} - \frac{1}{\sinh u} \right]$$

$$- \frac{2 D_1^2 \sinh u}{a \sin^3 v \sqrt{(\sinh^2 u + \sin^2 v)}} +$$

$$\frac{D_1^2}{a \sinh^6 u} \left[\frac{3}{2 \sinh u} \sin^{-1} \left(\frac{\sinh u}{\sqrt{\sinh^2 u + \sin^2 v}} \right) - \frac{\sin v}{2(\sinh^2 u + \sin^2 v)} - \frac{1}{\sin v} \right]$$

$$- \frac{2 D_1^2 \sin v}{a \sinh^3 u \sqrt{(\sinh^2 u + \sin^2 v)}}$$

$$+ \frac{C_1 D_1}{a} \left[\frac{\sinh u . \operatorname{Cot} v\ (\sin^2 v - \sinh^2 u)}{(\sinh^2 u + \sin^2 v)^3} \right] w \qquad \ldots (11)$$

Case II

(i) When $q_v = 0$, In this case, the self-superposable flows may be

$$q_u = \frac{A_1 V_1 W_1}{\sinh u \sin v \sqrt{\sinh^2 u + \sin^2 v}}$$

$$q_v = 0$$

$$q_v = \frac{B_1 U_2 V_2}{(\sinh^2 u + \sin^2 v)} \qquad \ldots (12)$$

and

$$\theta = - \frac{A_1 B_1 V_1 V_2 W_1'}{a \sin^3 v} \int \frac{U_2\ du}{\sinh^3 u\ (\sinh u + \sin^2 v)}$$

$$+ \frac{B_1^2 V_2^2}{a \sin v} \int \frac{U_2^2\ (\sin^2 v - \sinh^2 u) \operatorname{Cosh} u\ dU}{\sinh^2 u\ (\sinh^2 u + \sin^2 v)^3}$$

$$+ \frac{B_1^2 V_2^2}{a \sin v} \int \frac{U_2' U_2\ du}{\sinh u\ (\sinh^2 u + \sin^2 v)^2}$$

$$+ \frac{B_1^2 U_2^2}{a \sinh u} \int \frac{V_2^2 \operatorname{Cos} v\ (\sinh^2 - \sin^2 v)}{\sin^2 v\ (\sinh^2 u + \sin^2 v)^3} dv$$

$$+ \frac{B_1^2 U_2^2}{a \sinh u} \int \frac{V_2 V'_2\ dv}{\sin v (\sinh^2 u + \sin^2 v)^2}$$

$$+ \quad \frac{A_1^2 W_1^2}{a \sinh^2 u} \int \frac{(V_1 V_1' \operatorname{Cosec} v - V_1^2 \operatorname{cosec} v \cot v)}{\sin v(\sinh^2 u + \sin^2 v)} dv$$

$$+ \quad \frac{A_1^2 V_1^2}{a \sinh^2 u \sin^2 v (\sinh^2 u + \sin^2 v)^2} \int W_1 W_1' dw$$

$$- \frac{U'_2 \sinh u}{(\sinh^2 u + \sin^2 v)} \quad] \int W_1 \, dw \qquad .... (13)$$

(ii)

$$q_u = \frac{A_1 \operatorname{Cos} w}{\sinh u \sqrt{(\sinh^2 u + \sin^2 v)}}$$

$$q_v = 0 \qquad ... (14)$$

$$q_w = \frac{A_1 \sinh u . \sin v}{(\sinh_u^2 + \sin^2 v)}$$

and

$$\theta = - \frac{A_1^2 \sin w}{a \sin^3 v} \left[\operatorname{Coth} u - \frac{2}{\sin^2 v} \tan^{-1} \left(\frac{\tanh u}{\tan v} \right) \right]$$

$$+ \quad \frac{A_1^2 \sin v}{a} \left[\frac{2 \sinh u}{(\sinh^2 u + \sin^2 v)} - \frac{(2 \sin^2 v + 1)}{\sin v} \right.$$

$$\sin^{-1} \left(\frac{v}{\sqrt{\sinh^2 + \sin^2 v}} \right) - \frac{1}{4 \sin^3 v} \sin \left(\frac{2 \sin v}{\sqrt{\sinh^2 u + \sin^2 v}} \right)$$

$$+ \quad \frac{A_1^2}{a} \operatorname{Cosec} v - \log \left(\frac{\sinh^2 u}{\sqrt{\sinh^2 u + \sin^2 v}} \right)$$

$$\sin^{-1}\left(\frac{u}{\sqrt{\mathrm{Sinh}^2+\sin^2 v}}\right) - \frac{1}{4\sinh^3 u}\sin\left(\frac{2\sinh u}{\sqrt{\sinh^2 u+\sin^2 v}}\right)$$

$$+\frac{A_1^2}{2a}\ \mathrm{cosech}^2 u\left[\frac{\sinh u.\sin v}{(\sinh^2+\sin^2 v)} - \sin^{-1}\left(\frac{\sinh u}{\sqrt{\mathrm{Sinh}^2 u+\sin^2 v}}\right)\right]$$

$$-\frac{A_1^2}{2a}\ \frac{\sin^2 w}{\sinh^2 u(\sinh^2 u+\sin^2 v)^2}$$

$$-\frac{2A_1^2}{a}\ \frac{\sin^4 v\ \mathrm{Cosh}\ u\sin w}{(\sinh^2 h+\sin^2)^3} \qquad \ldots (15)$$

(iii)

$$q_u = \frac{C_2}{\sinh u\sin v\ \ \sinh^2 u+\sin^2 v}$$

$$q_v = 0$$

$$q_w = \frac{2}{(\sinh^2 u+\sin^2 v)} \qquad \ldots (16)$$

and

$$\theta = \frac{D_2^2}{a\sin v}\left[\mathrm{Cosech}^7 u\ \{\sin v.\ \mathrm{Cosech}\ u\right.$$

$$-\frac{1}{32}\sin\left(4\sin^{-1}\left(\frac{\sin v}{\sqrt{\sinh^2 u+\sin^2 v}}\right)\right)$$

$$-\frac{\sin v.\sinh u}{(\sinh^2 u+\sin^2 v} - \frac{(9+8\sin^4 v)}{8}\sin^{-1}\left(\frac{\sin v}{\sqrt{\sinh^2 u+\sin^2 v}}\right)$$

$$\left.-\frac{\sinh u}{\sin^2 v(\sinh^2 u+\sin^2 v)}\right]$$

$$+\frac{D_2^2}{a\sinh u}\left[\mathrm{Cosech}^7 u\ \{\sinh u.\ \mathrm{Cosec}\ v\right.$$

$$-\frac{1}{32}\sin\left(4\sin^{-1}\left(\frac{\sinh u}{\sqrt{\sinh^2 u+\sin^2 v}}\right)\right)$$

$$-\frac{\sin v.\ \sinh u}{(\sinh^2 u+\sin^2 v)}-\frac{(9+8\sinh^2 u)}{8}\sin^{-1}\left(\frac{\sinh u}{\sqrt{\sinh^2 u+\sin^2 v}}\right)$$

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$$-\frac{\sin v}{\sinh^2 u\,(\sinh^2 u+\sin^2 v)}\ ]$$

$$+\frac{C_2D_2\sin v.\coth u\,(\sin^2 v-\sinh^2 u)\,w}{(\sinh^2 u+\sin^2 v)} \quad .... (17)$$

Case III :- When $q_w = 0$, some self-superposable flow may be:

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(i) $$q_u = \frac{A_2\,V_3\,W_2}{\sinh u.\sin v\sqrt{\sinh^2 u+\sin^2 v}}$$

$$q_v = \frac{B_2\,U_3\,W_3}{\sinh u.\sin v\sqrt{\sinh^2 i+\sin^2 v}}$$

$$q_w = 0 \quad ... (18)$$

and

$$\theta = \frac{B_2^2\,W_3^2}{a\sin^2 v}\int\frac{(U_3'\,U_3\sinh u-U_3^2\cosh u)\,du}{\sinh^3 u\,(\sinh^2 u+\sin^2 v)^{3/2}}$$

$$-\frac{A_2B_2W_3W_2}{a\sin^3 u}\{V_3'\sin v-V_3\cos v\}\int\frac{U_3\,\mathrm{Cosech}^2 u\ \ du}{(\sinh^2 u+\sin^2 v)^{3/2}}$$

$$- \frac{A_2B_2W_3W_2}{a \sin^3 v} \{V_3' \sin v - V_3 \text{Cos } v\} \int \frac{U_3 \text{Cosech}^2 u \quad du}{(\sinh^2 u + \sin^2 v)^{3/2}}$$

$$- \frac{A_2B_2W_3W_3}{a \sin^3 v} \{U_3' \sinh u - U_3 \text{Cosh } u\} \int \frac{V_3 \text{ Cosec}^2 v \quad dv}{(\sinh^2 u + \sin^2 v)^{3/2}}$$

$$+ \frac{A_2^2 W_2^2}{a \sinh^2 u} \int \frac{(V_3' V_3 \sin v - V_3^2 \text{ Cos } v)}{\sin^3 v \, (\sinh^2 u + \sin^2 v)^{3/2}} dv$$

$$+ \frac{A_2^2 V_3^2}{a \sinh^2 \sin^2 v \, (\sinh^2 u + \sin^2 v)} \int W_2 W_2' \, dw$$

$$+ \frac{B_2^2 U_3^2}{a \sinh^2 u \sin^2 v \, (\sinh^2 u + \sin^2 v)} \int W_3 W_3' \, dw$$

... (19)

(ii)

$$q_u = \frac{A_2 \text{ Cos } w}{\sinh u \sqrt{\sinh^2 u + \sin^2 v}}$$

$$q_v = \frac{A_2 \text{ Cos } w}{\sin v \sqrt{\sinh^2 u + \sin^2 v}} \qquad \text{... (20)}$$

$$q_w = 0$$

and

$$\theta = \frac{A_2^2}{2a} \text{Cosech}^2 u . \text{ cosec}^2 v . \cos^2 w \qquad \text{.... (21)}$$

(iii)

$$q_u = \frac{C_3}{\sinh u \quad \sinh^2 u + \sin^2 v)} \qquad \text{... (22)}$$

$$q_v = \frac{D_3}{\sin v \, (\sinh^2 u + \sin^2 v)}$$

$$q_w = 0$$

$$\theta = \text{Constt.} \qquad \ldots (23)$$

Case IV

When $q_u = q_v = 0$ a possible solution of equation (3) is given by

$$q_u = 0$$

$$q_u = 0$$

$$q_w = \frac{A_3\ U_u\ V_u}{(\sinh^2 u + \sin^2 v)} \qquad \ldots (24)$$

Case V

When $q_v = 0$, $q_w = 0$, the self superposable flow is given by

$$q_u = \frac{A_4\ V_5\ W_4}{\sinh u . \sin v \sqrt{\sinh^2 u + \sin^2 v}}$$

$$q_v = 0 \qquad \ldots (25)$$

$$q_w = 0$$

Case VI

When $q_w = 0$, $q_u = 0$, the flow is

$$q_u = 0$$

$$q_v = \frac{A_5\ U_5\ U_5}{\sinh u . \sin v \sqrt{\sinh^2 u + \sin^2 v}} \qquad \ldots (26)$$

$$q_w = 0$$

In all the above cases U_n, V_n, W_n (n = 1,2,3,4 .....) are integrable functions of u, v and w respectively and A_n, B_n, C_n, D_n (n=1, 2, 3,....) are constants whch may be determined by boundary conditions.

SUPERPOSABLE FLUID MOTION

It has already been shown that the hydrodynamic flows given by equations (5) and (25) are self-superposable. It can also easily be shown that if, $\overline{q}_1$ and $\overline{q}_2$ be the two flows given by equation (5) and (25), then $\overline{p}$ for $\overline{q}_1 \pm \overline{q}_2$ can also be represented by the gradient of scalar quantities. Thus $\overline{q}_1$ and $\overline{q}_2$ will be mutually superposable and a flow

$$q_u = \frac{A\ V(v)\ W(w)}{\sinh u.\sin v\ \sqrt{\sinh^2 u + \sin^2 v}} \qquad \ldots (27)$$

$$q_v = \frac{B\ U(u)\ W(w)}{\sinh u.\ \sin v\sqrt{\sinh^2 u+\sin^2 v}}$$

$$q_w = \frac{C\ U(u).\ V(v)}{(\sinh^2 u + \sin^2 v)}$$

is possible. The same same flow can be determined by mutually superposing the flows (12) and (26) and (18) and (24).

PRESSURE DISTRIBUTION

It is interesting to note that 0 is nothing but Bernoulli function given by (4)

$$\theta = (\frac{q^2}{2\ g}) + h + \frac{p}{g} \qquad \ldots (28)$$

where q, g, h and p denote velocity, acceleration due to gravity, height above some hoprizontal plane of reference and the pressure head.

It is a well known fact that for an incompressible fluid the pressure head p is given by [4]

$$p = \frac{P}{\rho_o} + \text{constant.}$$

where P is the pressure distribution.

Also if the motion of the fluid be steady and slow then the value of h can be taken, without much loss of generality as u for the flows (5), (8) and (10); as v for the flow (12), (14) and (16); and w for (18), (20) and (22).

Thus for the flow (20) the pressure distribution

$$= K_1+K_2w + K_3\text{Cos}^2w \frac{(\text{Cosech}^2u + \text{Cose}^2v)}{(\sinh^2u + \sin^2v)}$$

$$+ K_4 \text{ Cosech}^2u \text{ cosec}^2v \cos^2w \qquad \ldots (29)$$

Similarly for the flow (22) taking $C_3 = D_3$

we have

$$P = K_5 + K_6w + K_7 \frac{(\text{cosech}^2u + \text{cosec}^2 v)}{(\sinh^2u + \sin^2v)} \ldots (30)$$

where K_1, K_2, K_3, K_4, K_5, K_6, K_7 are constants.

Similarly the pressure distribution for the other flows can be determined.

VORTICITY OF THE FLOW

It was shown by Ballabh [3] that for a self-superposable flow, verticity is constant along its stream lines. If $\bar{T}$ is a unit tangent along a stream line then

$$T \times q = 0 \qquad \ldots (31)$$

By equation (16) and (31) it can readily be shown that

$$T = \left[\frac{\sinh^2u + \sin^2v}{\{\sin^2v(\sinh^4 + \sinh^2u) + \sinh^4u + \sin^4v\}^{1/2}}, 0, \frac{\sinh^2 . \sin v}{\{\sin^2v(\sinh^4u+\sinh^2u) + \sinh^4u+\sin^4v\}^{1/2}} \right] \qquad \ldots (32)$$

Hence the vorticity of the flow (16) is constant along the curve represented by equation (32). Similarly the curves of constant velocity can also be found for other flows,

IRROTATIONALITY

Vorticity $\bar{G}$ for the flow (10) can be calculated

$$\bar{G} = \frac{D_1}{a \sin v(\sinh^2 u+\sin^2 v)^{1/2}} \left\{ \frac{(\sinh^2 u-\sin^2 u) \operatorname{Cos} v}{(\sinh^2 u + \sin^2 v)^2} \right\} \bar{e}_1$$

$$- \frac{D_1}{a \sinh u(\sinh^2 u+\sin^2 v)^{1/2}} \left\{ \frac{\cosh u \, (\sin^2 v-\sinh^2 u)}{(\sinh^2 u + \sin^2 v)^2} \right\} \bar{e}_2$$

$$-C_1 \frac{\operatorname{Cosech} u. \operatorname{Coth} u. \operatorname{Cosec} v}{a \, (\sinh^2 u+\sin^2 v)} \bar{e}_3 \qquad \ldots. \; (33)$$

It is clear from equation (33) that flow (10) is not irrotational. For the flow (22)

$$\bar{G} = 0.$$

Hence the flow (22) is irrotational throughout.

Similar conclusions can be drawn for the other flow discussed earlier.

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EFFECT OF NON-HOMOGENEITY OF CRITICAL LOAD IN BUCKLING OF A BAR UNDER DISTRIBUTED AXIAL LOAD

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ABSTRACT

Critical load for buckling of a bar of non-homogeneous material has been obtained by two methods : (i) by using the differential equation of the deflection curve and (ii) by the energy method. The non-homogeneity of the bar has been assumed to be arising out of the variation of Young's modulus of the bar with the longitudinal distance. Critical loads obtained by the methods are found to be nearly same. The effect of non-homogeneity of this type has been observed to increase the critical load of the associated homogeneous case.

1. INTRODUCTION

The beam problem is very well-known problem in the theory of elasticity and it has significant importance in engineering structures. Thus from structural point of view, knowledge of critical buckling load of such beams is essential. The critical loads for buckling for a beam of homogeneous material under different conditions have been obtained by many investigators (Timoshenko and Gere [2]). But if the material is non-homogeneous a complete solution of the governing differential equation may not always be

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possible and in such a case approximate methods e.g. energy method, other variational methods or purturbation method, are to be used to get approximate critical load for buckling.

Our present investigation aims at the effect of non-homogeneity on critical load for buckling of a bar whose Young's modulus varies Linearly with its longitudinal distance. The Origin of co-ordinates has been suitably chosen in order to simplify the governing differential equation of the deflection curve. Solving this differential equation and using necessary and conditions critical load for buckling has been obtained. Critical load for buckling has also been obtained by assuming an approximate shape of the deflection curve and then by using the energy method. As is expected, (Timoshenko [1]) the critical load found by the energy method is slightly grater than that found by the first method. Of course, by increasing the number of parameters in the approximate equation for the deflection curve the difference of two critical load values may be made sufficiently small. In this paper it has been observed that critical load in the non-homogeneous case is greater than that in the associated homogeneous case.

2. FORMULATION OF THE PROBLEM

Let us consider a bar of length l whose lower end is vertically built in, upper end is free and the weight is uniformly distributed along the length.

We choose the co-ordinate axes as shown in the figure. If the bar buckles as shown by the dotted line in the figure, the differential equation of the curve is

$$EI \frac{d^2y}{dx^2} = \int_x^b q\,(\eta - y)\,d\xi \qquad \ldots (1)$$

where $b = a + \ell$, $a = OA$ (in the figure), and the integral on the right hand side of (1) represents the bending moment of any cross section fg produced by the uniform distributed load of intensity q.

Differentiation of (1) with respect to x gives

$$\frac{d}{dx}\left[EI\,\frac{d^2y}{dx^2}\right] = -q\,(b-x)\,\frac{dy}{dx} \qquad \dots (2)$$

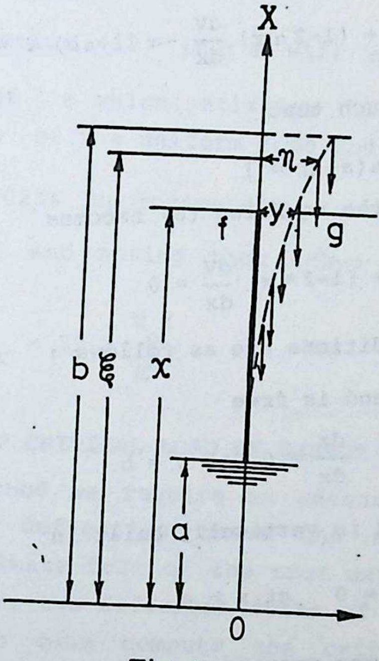

Fig.1

We now assume that the Young's modulus E of the bar obeys the law

$$E = E_o\left(\frac{x}{a}\right),\ a \le x \le b \qquad \dots (3)$$

Where E_o is the value of E at the lower end of the bar and a is a suitable length to be defined latter.

Using (3) and writing $z = \frac{dy}{dx}$, equation (2) becomes where

$$x \frac{d^2 z}{dx^2} + \frac{dz}{dx} + \alpha^2 (b-x) z = 0 \qquad \ldots (4)$$

$$\alpha^2 = \frac{qa}{E_o I} \qquad \ldots (5)$$

Setting $z = e^{-\alpha x}.V$, equation (4) becomes

$$x \frac{d^2 V}{dx^2} + (1-2\alpha x) \frac{dV}{dx} - \alpha (1-\alpha b) V = 0 \qquad \ldots (6)$$

we now choose a such that

$$\alpha b = \alpha(a+\ell) = 1$$

with this choice the equation (6) becomes

$$x \frac{d^2 V}{dx^2} + (1-2\alpha x) \frac{dV}{dx} = 0 \qquad \ldots (8)$$

The boundary conditions are as follows :

Since the upper end is free

$$\frac{d^2 V}{dx^2} = \frac{dz}{dx} = 0 \quad \text{at } x = b$$

and the lower end is vertically built in

$$\frac{dy}{dx} = z = 0 \quad \text{at } x = a \qquad \ldots (9b)$$

3. SOLUTION OF THE PROBLEM

The solution of (8) is

$$V = A\, H(x) + B$$

Where $H(x) = \int \frac{e^{2\alpha x}}{x} dx$

and A and B are constants.

Thus $z = e^{-\alpha x}. [A\, H(x) + B]$ ... (10)

Using boundary condition (9a) we get

$$z = A e^{-\alpha x}[H(x)-H(b)+e^2] \qquad \ldots (11)$$

Where (7) has been used.

Condition 9(b) gives

$$\int_a^b \frac{e^{2\alpha x}}{x}\, dx = e^2$$

which by (7) becomes

$$\int_{2\alpha a}^{2} \frac{e^t}{t}\, dt = e^2 \qquad \ldots (12)$$

The roots of equation (12) in α a will make z=0 at x=a The lowest value of a which satisfies (12) corresponds to the critical value of the uniform load and is found to be

$$\alpha a = 0.0234 \text{ approximately.}$$

Using (7) and (5) and noting that $\ell = b-a$ we compute the critical load as

$$(q\ell)_{cr} = 39.8 \frac{E I_o}{\ell^2} \qquad \ldots (13)$$

4. COMPUTATION OF CRITICAL LOAD BY ENERGY METHOD

In this method we require an advance knowledge of the shape of the deflection curve and so we have to consider an approximate form of the true deflection curve. Here we shall assume two different forms of the deflection curve and in each case compute the critical load for buckling. Since the critical load for true deflection curve is minimum, comparing the above two values of critical loads we can select the better of the two approximate curves.

Let the approximate deflection curve be y = f (x) Here we consider the following two forms of f(x).

4.1 f(x) is a Tribonometric Function of x

Here we approximate the true deflection curve by taking

$$y = \delta_1(1-\cos\frac{\pi}{2}X) + \delta_2(1-\mathrm{Cos}\frac{3\pi}{2}X) + \delta_3(1-\mathrm{Cos}\frac{5\pi}{2}X) \quad \ldots (14)$$

where $(b-a)\,X = x-a$ ... (15)

and δ_i $(i = 1,2,3)$ are parameters.

The deflection y in (14) then satisfies the end conditions (9a) and (9b).

The bending moment at any cross section fg in the figure is

$$M = \int_x^b q(\eta-y)\,d\xi = q(b-a)\sum_{r=1}^{3}\delta_r\,S_r$$

Where $S_r = (1-X)\,\mathrm{Cos}\,(\theta_r X) + \frac{1}{\theta_r}[(-1)^r+\mathrm{Sin}(\theta_r X)]$

and $\theta_r = \frac{\pi}{2}(2r-1)$

The strain energy of bending is given by [Timoshenko and Gere [2] (1961), p.105]

$$\Delta U = \int_a^b \frac{M^2}{2EI}\,dx = \frac{q^2(b-a)^3}{2\,E_oI}\sum_{r,s=1}^{3} I_{rs}\,\delta_r\,\delta_s$$

Where $I_{rs} = \int_0^1 S_r S_s\,F(X)\,dX$ ... (16)

and $F(X) = 1/[1+(b/a-1)\,X]$

In view of (14) the total work produced by the load during buckling is, [Timoshenko and Gere [2] (1961), p.108]

$$\Delta T = \frac{1}{2}q\int_a^b (b-x)\left(\frac{dy}{dx}\right)^2 dx = \frac{\pi^2}{8}\,q\sum_{r,s=1}^{3}\beta_{rs}\,\delta_r\,\delta_s \ldots (17)$$

where $\beta_{rr} = \frac{r^2}{4} - \frac{1}{\pi^2}$, $r = 1,2,3$

$$\beta_{12} = \frac{3}{\pi^2}, \quad \beta_{13} = -\frac{5}{\pi^2}, \quad \beta_{23} = \frac{15}{\pi^2}, \quad \beta_{rs} = \beta_{sr}$$

The critical value of the axial load satisfies the equation, [Timoshenko and Gere [2] (1961), p.84'

$$\Delta U = \Delta T \qquad \ldots (18)$$

Substituting (16) and (17) in (18) we get

$$(q)_{cr} = P\Big(\sum_{r,s=1}^{3} \beta_{rs}\,\delta_r\,\delta_s\Big) \Big/ \sum_{r,s=1}^{3} I_{rs}\,\delta_r\,\delta_s \qquad \ldots (19)$$

where $P = \frac{\pi^2 E_o I}{4\ell^3}$

We now choose parameters δ_r $(r=1,2,3)$ such that $(q)_{cr}$ becomes minimum. The conditions for that are

$$\frac{\partial}{\partial \delta_r}(q)_{cr} = 0, \; r=1,2,3. \qquad \ldots (20)$$

From (19) and (20) and from the fact that the possibility of buckling occurs when δ_r $(r=1,2,3) \neq 0$, we get

$$\det(a_{rs}) = 0, \; r, s = 1,2,3 \qquad \ldots (21)$$

where

$$a_{rs} = \beta_{rs} - \frac{(q)_{cr}}{P} I_{rs}, \; r,s = 1,2,3. \qquad \ldots (22)$$

Now using (7) we find that

$$\frac{(q)_{cr}}{P} \quad \frac{4(u-1)^3}{\pi^2 u^2} = h(u) \qquad \ldots (23)$$

where $u = \frac{b}{a}$

Substituting (23) in (22) we get a_{rs} as functions of b/a. Equation (21) then gives the value of b/a corresponding to the critical load.

Solving (21) we obtain

$$\frac{a}{b} = 0.0228 \text{ approximately,}$$

Using (7) and noting that $\ell = b-a$ we compute the critical load as

$$(q\ell)_{cr} = 40.92 \frac{E_o I}{\ell^2} \qquad \ldots (24)$$

Comparing (13) with (24) we find that with the approximate deflection curve given by (14) the error in the value of the critical load by the energy method is less than 2.8 percent. It may be mentioned here that if we put $\delta_3=0$ in (14) or in otherwise if there are only two parameters in the deflection curve the error in the critical load was observed to be 7.26 percent. Instead of (14) had we assumed

$$Y = \sum_{k=1}^{n} \delta_k [1-\text{Cos } (2k-1)\frac{\pi}{2} x]$$

we could have obtained a value for the critical load very close to that given by (13) for sufficiently large value of n.

4.2 f(x) is a Polynomial in x.

Here the deflection curve is assumed as

$$Y = \delta_1 x + \delta_2 x^2 + \delta_3 x^3 \qquad \ldots (25)$$

So that the end conditions (9a) and (9b) are satisfied.

Using (25) and proceeding as in 4.1 we obtain equation (19), in which

$P = 288 E_o I k^5/\ell^3$,

$k = (b/a)-1$,

$$I_{11} = 36h_1,$$

$$I_{22} = [16(k-z)^2 h_1 \pm 64(k-z)h_2 \pm 64h_3]/k^2,$$

$$I_{33} = [\,9(k^4-4k^3 + 10k^2 -12k+9)h_1 + 36(k^3-5k^2+9k-9)h_2 + 18(5k^2-18k + 27)h_3 + 54(k-6)h_4+81h_5/k^4,$$

$$I_{12} = [24\,(k-2)h_1 + 48h_2]/k,$$

$$I_{23} = [12\,(k^3-4k^2 + 7k-6)h_1+ 24\,(2k^2-7k+9)h_2 + 12(7k-18)h_3 + 72h_4]/k^3,$$

$$I_{31} = 18\,[(k^2-2k+3)h_1 + (2k-6)h_2 + 3h_3]/k^2,$$

$$\beta_{11} = \frac{1}{4},\ \beta_{22}= \frac{1}{6},\ \beta_{33} = \frac{3}{20},\ \beta_{12} = \frac{1}{6},$$

$$\beta_{23} = \frac{3}{20},\ \beta_{13} = \frac{1}{8},\ \beta_{rs} = \beta_{sr} \text{ and } I_{rs} = I_{sr} = (r,s=1,2,3)$$

$$h_1 = [u^4(12 \log u + 23) - 48ku^3-36u^2+16u-3)/12,$$

$$h_2 = [-4u^5+5ku^4+10u^3-10u^2+5u-1]/5,$$

$$h_3 = [u^6-15u^4 + 40u^3-45u^3 + 24u-5]/30,$$

$$h_4 = [u^7-35u^4 + 105u^3-126u^2 + 70u - 15]/105,$$

$$h_5 = [u^8-70u^4 + 224u^3-280u^2 + 160u - 35]/280.$$

Now, choosing papameters δ_r (r=1,2,3) such that $(q)_{cr}$ is minimum and using similar arguments as in the previous case we obtain the value of a/b corresponding to the critical load as

$$\frac{a}{b} = 0.0210 \text{ approximately.}$$

Then by (7) the critical load is obtained as

$$(q\ell)_{cr} = 44.59\,\frac{E_o I}{\ell^2} \qquad \dots (26)$$

Comparison of equation (26) with (24) and (13) suggests that equation (14) gives a closer approximation to the true deflection curve than that given by equation (25). Moreover, the approximate curve with two parameters (δ_3=0) in (14) is even better than that with three parameters in (25).

5. CONCLUSION

The critical load for buckling in the associated homogeneous case is given by [Timoshenko and Gere, [2] p.103]

$$(q\ell)_{cr} = 7.837 \frac{E_o I}{\ell^2}$$

Comparison of this value with out present value given by (13) shows that the effect of non-homogeneity as given by (3) is to increase the critical load to about 5.08 times.

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GEOPHYSICAL PROSPECTING WITH LIGHTNING RADIATION

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ABSTRACT

A new method of geophysical prospecting using the lightning radiation is proposed. The detailed theoretical formulation shows that the ground conductivity profile can be plotted by recording the values of ratio of vertical to horizontal component of electrical field radiated from nearby lightning discharge (at a distance of 50 to 100 kms). This profile may be used for miniral exploration.

INTRODUCTION :

Use of natural fields for determination of ground conductivity dates back to 1953 when Cagniard [1] showed how the ratio of alternating magnetic field to alternating electric field plotted as a function of frequency can yield information about the variation of resistivity with depth within the earth. The method known as MT method measures simultaneously the magnetic and electrical oscillations caused by ionospheric currents and thunderstorms. These fields are recorded at different frequencies in the range from less than one hertz to 10 hertz. The same method using higher frequencies in audio range is known as AMT method.

Another closely related method was introduced in 1958 when Ward [2] measured the tilt of magnetic component of natural electromagnetic field of distant thunderstorms (at thousand of kms). At these large distances the field propagates in such a way that electric components are almost vertical and magnetic components almost hiorzontal. Presence of a conducting zone with

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a vertical boundary tilts the plane of polarisation and produces an appreciable vertical component of magnetic field. The tilt in (inclination of) the resultant magnetic field is measure of discountinuity in resistivity. This is known as AFMAG method as it uses the characteristics of audio fequency magnetic fields.

Another method on similar principle but using an artificially transmitted electromegnetic radiation is conductivity determination by wave-tilt measurements [3] where conductivity is determined from the ratio of horizontal component (directly measured) and vertical component of electrical field (determined by measuring horizontal magnetic field) emitted from VLF transmitter at a known frequency, power and distance from point of observation. Recently artificial source has been used for AMT method also [4].

These methods determine the ground conductivity and the data is interpreted according to required applications, e.g. search of metals and minirals or finding the depth of bedrock or in geothermal exploration. The first of these methods uses radiation from the source at unknown distance. The magnitude of the variation is indefinite. For long distance sources high sensitivity is needed. A near thunderstorm would disturb the conditions and even the instrument may be damaged. Moreover, the theory is not applicable to for the radiation from near thunderstorm or transmission line. This condition is also same for AFMAG method, though this method is more qualitative than quantitative. The wave-tilt measurements from artificial transmitter at known distance frequency and power are better but they too may be disturbed by thunderstorms and the theory uses many approximations [5].

In this paper a possibility of new method of geophysical prospecting using natural source is analysed and its use for different purposes is proposed.

LIGHTNING AS NATURAL TRANSMITTER :

Lightning is a heavy current transient electric discharge between two opposite charge centres within a thunderstorm or between lower charge of a thunderstorm and the ground. It results in a transfer of charge from one place to another in form of heavy currents with peak value of order of few tens of killo amperes. This time varying current through air column flows for finite time (∼ 200 μs) and produces an oscillating dipole consisting of the current channel of varying length and its image in finitely conducting ground. Experimentally observed radiation from lightning extends over whole range of electromagnetic radiation beginning from few hertz in ELF to UHF and microwave besides the visible and infrared radiation. The maximum power in radio wave range is in VLF region and peaks between 5 kHz and 10 kHz. Thus the lightning is a strong natural source of VLF radiation. This VIF radiation can be used for geophysical prospecting if the details of the influence of ground conuctivity on the radiated fields is worked out. This is discussed in next section.

GROUND CONDUCTIVITY DEPENDENCE OF LIGHTNING RADIATION :

Pathak et al. [6] showed that the VLF radiation from lightning may be accounted for by radiation emitted from an oscillating dipole consisting of lightning channel and its image in the ground. They calculated the electric and magnetic field components by formulating a vector potential associated with lightning channel and an image vector potential associated with the image of the channel. The image vector potential and therefore the resultant vertical fields are functions of ground conductivity. It can be shown [7] that the radiation from a return stroke (vertical channel from ground to cloud) is independent of ground conductivity while radiation from other strokes like K-changes (vertical or slightly inclined channel within the cloud) or horizontal discharges depends largely on ground conductivity. Physically these results are understood from the following well known facts: (i) A current channel (with any arbitrary orientation) over a perfectly conducting ground (hypothetical case) produces only vertical electrical fields (ii) An imperfectly conducting ground develops a horizontal component of electric field in two ways : (a) Horizontal currents perpendicular to the plane of incidence (a vertical plane containing source and receiver) produce a horizontal E perpendicular to the plane of incidence. (b) Vertical currents at sufficiently great heights and horizontal currents in the plane of incidence produce a horizontal E parallel to the plane of incidence. Thus the return stroke does not produce horizontal electric component and therefore the vertical component from return stroke remains independent of ground conductivity. In other cases like K-change and horizontal channel, a horizontal component is developed because of penetration of a part of energy into the earth and making reflection coefficient less than unity. As the reflection coefficient, hence the penetration depends upon the ground conductivity, the horizontal and vertical E field also become the functions of conductivity.

To obtain the analytical expressions for vertical and horizontal electric fields a vector potential **A** is formulated from the channel current I as follows:

$$\mathbf{A}(t)=\frac{\mu_o}{4\pi}\int\frac{J(t')}{r}\,dv' = \frac{\mu_o}{4\pi}\int\frac{I(t')}{r}\,\mathbf{dz}' = \frac{\mu_o}{4\pi}\frac{o}{r}F(t') \quad ...(1)$$

where t' $(= t - \frac{r}{c})$ is rerarded time, **dz'** is retarded elemental length along the channel and A_o a vector constant in time. The distance of observation point being larger than the channel dimensions is kept outside the integral. Assuming the origin of coordinate system at starting point of channel and z-axis vertically upward r, θ and ϕ- components of **A** can be written and electric and magnetic fields can be devaluated from the component equations of the relations

$$H = \frac{1}{\mu_o} \nabla \cdot X\, A \text{ and } E = \frac{1}{\varepsilon_o} \int \nabla X.H\, dt \qquad \ldots (2)$$

Similarly the field components due to image vector potential $A' = R_c A$, R_c being reflection coefficient, can be written and expressions for corresponding field components E_r'. E_θ'. E_ϕ' can be evaluated. But we need most commonly measured vertical and horizontal components of the fields. So the vertical electric field is obtained by summing all the vertical components of E_r, E_θ and E_r', E_θ', i.e.

$$E_v(t) = \frac{A_o}{4\pi\varepsilon_o} \left[\frac{f_1}{r^3} \int F(t')dt + \frac{f_1}{r^2 c} \frac{\partial F(t')}{\partial t} \right] \qquad \ldots (3)$$

For horizontal electric field, the horizontal components of E_r, E_θ, E_r, E_θ are added to obtain the field in the plane of incidence. The field perpendicular to the plane of incidence is the sum of E_ϕ and E_ϕ'. The resultant horizontal field is square root of the sum of the squares of fields in two perpendicular horizontal directions i.e.

$$E_h(t) = \frac{A_o}{4\pi\varepsilon_o} \Big[\Big\{ \frac{f_3}{r^3} \int F(t')dt + \frac{f_3}{r^2 c} F(t') + \frac{f_4}{rc^2} \frac{\partial}{\partial t} F(t')) \Big\}^2 + \Big\{ \frac{f_5}{r^3} \int F(t')dt + \frac{f_5}{r^2 c} F(t') + \frac{f_5}{rc^2} \frac{\partial}{\partial t} F(t') \Big\}^2 \Big]^{\frac{1}{2}}$$

$$\ldots (4)$$

where,

$$f_1=\cos\theta_1[2-\sin^2\theta+(3\sin^2\theta-2)R_V]-\sin\theta_1\sin\theta\cos\theta\cos(\theta_1-\phi)(1-3R'_V) \quad \ldots (5)$$

$$f_2=(1+R_V)[\cos\theta_1\sin^2\theta+\sin\theta_1\sin\theta\cos\theta\cos(\phi_1-\phi] \quad \ldots (6)$$

$$f_3=(1+3R_V)\cos\theta_1\sin\theta\cos\theta+\sin\theta_1[2-\cos^2\theta+(2-3\cos^2\theta)R_V]\cos(\phi_1-\phi) \quad \ldots (7)$$

$$f_4=(1-R_V)[-\cos\theta_1\sin\theta\cos\theta+\sin\theta_1\cos^2\theta\cos(\phi_1-\phi)] \quad \ldots (8)$$

$$f_5=-(1+R_h)\sin\theta_1\sin(\phi_1-\phi) \quad \ldots (9)$$

$$R_V=\frac{S-T}{S+T};\ R_h=\frac{\cos\psi-T}{\cos\psi+T} \quad \ldots (10)$$

with

$$S=[(\varepsilon+\frac{\sigma}{iw})/\varepsilon_0]\cos\psi;\ T=[(\varepsilon+\frac{\sigma}{iw})/\varepsilon_0-\sin^2\psi]^{\frac{1}{2}} \quad \ldots (11)$$

and $\psi=\pi-\theta$. The reflection coefficients R_V and R_h are the functions of permittivity and conductivity of the ground. Therefore the electric field components are also the functions of these parameters.

THE PARAMETER TO BE MEASURED

Equations (3) and (4) give time-dependent values of vertical and horizontal electric field. These can be subjected to Fourier transformations to get the frequency dependent values as

$$E_{v,h}(w) = \int_o^\infty E_{v,h}(t)e^{-iwt}dt \qquad ...(12)$$

However, the field-components thus obtained are not the functions of w and σ only. They also contain lightning parameters through A_o. Eq(1) shows that A_o would contain lightning constants like peak value of current and decay constants. These vary from one discharge to another and not easy to determine. Simple way to get rid of these parameters is to consider ratio $R = E_v/E_h$ which would depend only on orientation of lightning discharge besides w and σ. Influence of orientation of the channel is very easy to visualise through eqs. (5)-(9), as follows:

For vertical discharge $\theta_1 = 0$ then

$$f_1^v = 2 - \sin^2\theta + R_v(3\sin^2\theta - 2)$$

$$f_2^v = (1+R_v)\,\mathrm{Sin}^2\theta$$

$$f_3^v = (1+3R_v)\,\mathrm{Sin}\theta\cos\theta \qquad ...(12)$$

$$f_4^v = -(1-R_v)\sin\theta\cos\theta$$

$$f_5^v = 0$$

For a vertical discharge starting from ground like return stroke; $\theta = \pi/2$, then $f_1 = f_2 = (1+R_v)$, $f_3 = f_4 = f_5 = 0$. This gives $E_h^v = 0$ and $R = \infty$ in accordance with the concepts that a vertical current near ground does not produce any horizontal field. Also for this case $|R_v| = 1$, so E_v becomes independent of ground conductivity which is obvious when the horizontal component is not developed i.e. energy ponetration into the ground is zero. Same result is obtained by Pathak and Hazarika [7].

For vertical discharge at sufficient height h above the ground $\theta = \pi/2 + \sin^{-1}(\frac{h}{r})$. The computed values of R at a fixed frequency of 5 kHz are plotted with ground conductivity for different values of r in Fig.1a.

For horizontal discharge; $\theta_1 = \pi/2$. But the channel may have any arbitrary inclination from the plane of incidence. If the channel is perpendicular to the plane of incidence i.e. $_1 - = \pi/2$ so that $f_1 = f_2 = f_3 = f_4 = 0$ and $f_5^h = (1 + R_v)$, we have $E^h = 0$ and therefore $R = 0$. The field is confined to a plane perpendicular to the plane of incidence. For the case when channel is in the plane of incidence. $\phi_1 - \phi = 0$, so that

$$\begin{aligned}
f_1^h &= (1 - 3R_v) \sin\theta \cos\theta \\
f_2^h &= (1 + R_v) \sin\theta \cos\theta \\
f_3^h &= 2 - \cos^2\theta + (2 - 3\cos^2\theta)R_v \qquad \ldots (13) \\
f_4^h &= (1 - R_v) \cos^2\theta \\
f_5^h &= 0
\end{aligned}$$

This gives E_h^h confined to the plane of incidence as expected. Again $\theta = \pi/2 + \sin^{-1}(\frac{h}{r})$. The variation in the values of R is too small to be plotted or to be measured over the experimental error (see Table 1). The overall variation of R for arbitrary orientation of horizontal channel is between zero and 0.41. For requirement of high sensitivity and of determination of orientation $(\varphi_1 - \phi)$, the horizontal channel is not recomended for use in determination of ground conductivity.

FIELD PROCEDURE :

As shown above the parameter depending upon minimum number of lightning parameter is ratio R of vertical and horizontal electric field. A non-zero, finite value of R with sufficient variation is obtained only for the case of intracloud vertical lightning. Field procedure would be to measure E_v and E_h at a fixed frequency (say 5 kHz). These are measured with the help of vertical and horizontal whip antinnas suplimented by tuned amplifiers. Amplification is to be sufficient to amplify a signal of a fraction of mV to the magnitude required by recorder. For measuring the resultant horizontal field, signals from two crossed whips are fed into a system of squaring, adding and square-rooting IC's connected as shown in Fig.1b. After this stage E_v and E_h may be recorded separately and ratio may be obtained manually or a dividing IC may be used to obtain R directly. The distance of discharge is recorded from the time log Δt between electrical signal and thunder heard. Then $r = \Delta t$ X speed of sound. If $E_v > E_h$ i.e. $R > 1$, the value of σ may be determined from Fig. 1a. By using many amplifieres tuned to different frequencies, average σ for different depths of penetration for these frequencies is plotted. This gives the conductivity profle with depth. This may be used for exploring the presence of minirals within the earth, and the distribution of ground water.

CONCLUSION :

From the fargoing discussion it is clear that by the observation of ratio of vertical and horizontal electic field radiated from nearly lightning (in range of distance of 50 to 100 km) the ground conductivity profile may be determined for the use in miniral exploration. The expressions are not valid for very near large distances (more than 100 km).

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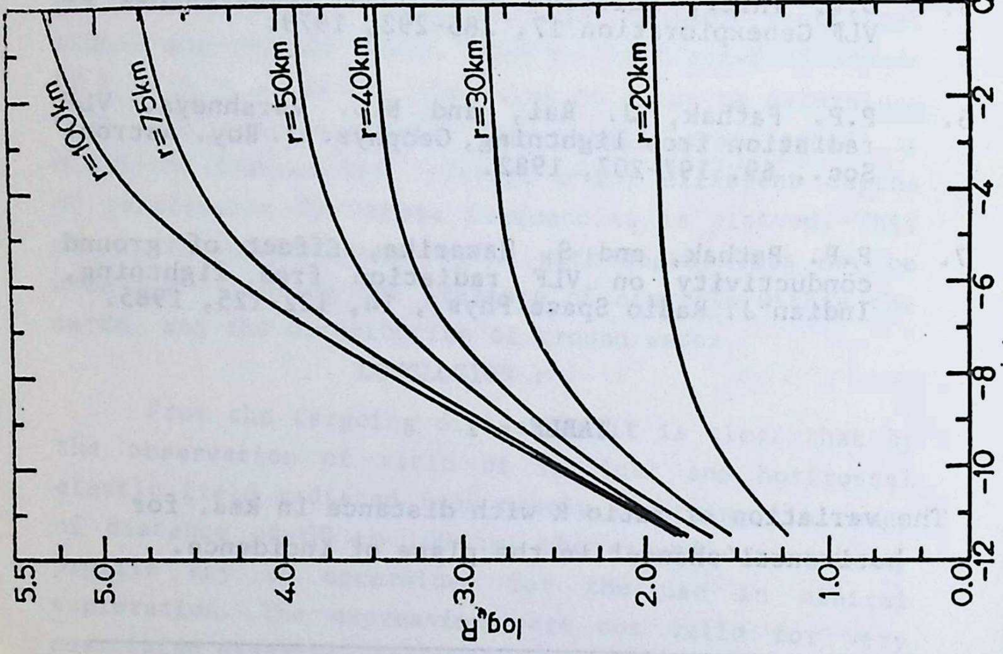
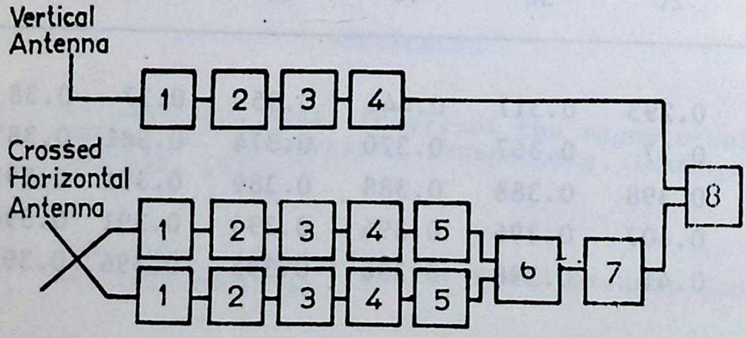
TABLE - 1

The variation of ratio R with distance in kms. for horizontal channel in the plane of incidence.

$\sigma(\Omega m)^{-1}$	20	30	40	50	75	100
10^{-5}	0.295	0.317	0.441	0.35	0.37	0.38
10^{-4}	0.37	0.367	0.370	0.374	0.381	0.385
10^{-3}	0.398	0.388	0.388	0.389	0.391	0.393
10^{-2}	0.407	0.396	0.394	0.394	0.391	0.396
10^{-1}	0.41	0.398	0.396	0.395	0.396	0.397




Vertical Antenna
Crossed Horizontal Antenna
1
2
3
4
5
6
7
8

1. Matching Amplifier (Unity gain), 2. Tuner , 3. Voltage Amplifier,
4. Power Amplifier, 5. Squaring V.A., 6. Adding Circuit,
7. Square Rooting V.A., 8. Dividing Circuit (Digital Recorder)

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NOTE ON THE DEFLECTION IN A SIMPLY SUPPORTED TIMOSHENKO BEAM OF NONHOMOGENEOUS MATERIAL

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ABSTRACT

In expressions for the deflections in a simply supported beam of nonhomogeneous material have been obtained by using both technical and Timoshenko beam theories. It has been assess that the Young's modulus of the beam is a function of position and the beam is under the action of variable load. A comparative study has been made, to assess the effect of nonhomogeneity as has been considered in our note, in the tables for different values of the nonhomogeneity parameter.

Keywords : Timoshenko beam, Deflection, Nonhomogeneous material, Young's modulus.

INTRODUCTION :

The one dimensional theory of beams may have wider range of applicability provided the effects of transverse shear deformation, and the effects of rotatory inertia in the case of vibrating beams, are considered. When these effects are taken into account in the technical theory of beams, the new theory is known as Timoshenko beam theory [17, 18]. Dengler and Goland [6], Goland et

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al.[8], Miklowitz [11,12] studied the problems of dynamic response of Timoshenko beams. Achenback and Sun [1] obtained the steady state solutions of the response of the Timoshenko beam to a force moving with constant velocity over the beam which was supported by a deformable foundation. Ahmadi and Hashemi [2] studied the problem of vibration of an infinitely long Timoshenko beam under two types of random loadings. Chang and Juan [3] derived a set of equations for the free vibration of an inclined bar with an end constraint including the effect of rotatory inertia and shear deformation. Chen [4] presented the general dynamic stiffness matrix of a Timoshenko beam for transverse vibration. Among the many other problems of the dynamic response of Timoshenko beams mention may be made of the interesting works of Howson and Williams [9], Mead [10], Nayfeh [13], Park [14], Rao [15], Rossettos and Squires [16]. The statical Timoshenko beam problems have also received considerable attention in the literature. A comparative study, for the deflections in a uniformily loaded simply supported elastically homogeneous beam following the above theories, has been given in Dym and Shames [7].

As we know that there are plenty of materials in nature which are not elastically homogeneous and as it has been experimentally verified that in most of the cases the elastic co-efficients should not be taken as uniform throughout the beam, the purpose of the present investigation is to extend the foregoing studies given in Dym and Shames [7] to a class of nonhomogeneous beams subjected to nonuniform loading. The nonhomogeneity has been supposed to be arising from the variation of

Young's modulus with position. The general expressions for the deflections have been given. Numerical evaluations of deflections at different positions of the beam have been carried out to study the effects of shear deformation at a glance. Results of the corresponding homogeneous cases indicate that nonhomogeneity has significant effect on deflection.

FORMULATION OF THE PROBLEM AND ITS SOLUTION

(a) Solution based on the technical theory of beams

Choosing x-axis along the axis of the beam, let the position of the beam be given by $a \leqslant x \leqslant 2a \; (a > 0)$. The equation of equilibrium for the bending of the beam is given by

$$\frac{d^2}{dx^2}\left[EI\frac{d^2w}{dx^2}\right] = qx^2 \qquad \dots (1)$$

where w represents the deflection of the centreline of the beam under the action of variable load q(x) on the centroidal axis of the beam. E is the variable Young's modulus and I is the moment of inertia of the cross section of the beam about an axis lying in its plane.

We assume that the Young's modulus of the beam is a function of position according to the law,

$$E = E_o\, e^{-m\left(\frac{x}{a}\right)} \qquad \dots (2)$$

and loading depends upon the position following the relation,

$$q = q_o \left(\frac{x}{a}\right)^n \qquad \text{... (3)}$$

where E_o, q_o, m and n are real constants.

As regards the boundary condition of the problem we assume that both ends of the beam are simply supported. Mathematically speaking,

$$w = 0$$

$$\text{and} \quad \frac{d^2w}{dx^2} = 0 \text{ at } x = a \text{ and } x = 2a. \qquad \text{... (4)}$$

Introducing the nondimensional variable $X = \frac{x}{a}$ and taking into account of (2) and (3), field equation (1) becomes,

$$\frac{E_oI}{a^4}\frac{d^2}{dX^2}\left[e^{-mX}\frac{d^2w}{dX^2}\right] = q_oX^n \qquad \text{... (5)}$$

and the boundary condition (4) takes the form,

$$w = 0$$

$$\text{and} \quad \frac{d^2w}{dX^2} = 0 \text{ at } X = 1 \text{ and } X = 2. \qquad \text{... (6)}$$

Thus our problem is to solve (5) subject to (6).

Integrating equation (5) twice with respect to X and making use of the boundary condition $\frac{d^2w}{dX^2} = 0$ at $X = 1$ and $X = 2$ we obtain,

$$\frac{d^2w}{dx^2} = A_1X^{n+2}e^{mX} + B_1Xe^{mX} + C_1e^{mX} \qquad \text{... (5a)}$$

finally integrating (5a) twice with respect to X and making use of the boundary condition $w=0$ at $X=1$ and $X=2$ we obtain the expression for w as

$$w = w_o(X-1) + A_1 f(X) + B_1 g(X) + C_1 \theta(X) \qquad \dots (7)$$

where

$$f(X) = \int_1^X \left(\int_1^u t^{n+2} e^{mt} dt \right) du$$

$$g(X) = \int_1^X \left(\int_1^u t\, e^{mt} dt \right) du$$

$$\theta(X) = \int_1^X \left(\int_1^u e^{mt} dt \right) du$$

$$A_1 = \frac{a^4}{E_o I} \frac{q_o}{(n+1)\ (n+2)}$$

$$B_1 = A_1 (1-2^{n+2})$$

$$C_1 = A_1 (2^{n+2}-2)$$

and w_o is the value of $\frac{dw}{dX}$ at $X=1$ and it satisfies the equation

$$w_o + A_1 f(2) + B_1 g(2) + C_1 \theta(2) = 0. \qquad \dots (8)$$

(b) Solution based on the Timoshenko beam theory

The method discussed in the previous section does not indclude the effects of shear deformation. For short stubby beams, however, it is natural that this contribution cannot be neglected. Hence arises, the question of considering Timoshenko theory of beams. In this theory it is assumed that line elements normal to the centreline of the beam in the undeformed state move only in the vertical direction and also remain vertical during deformation. Line elements tangent to centreline undergo a rotation β (x) and satisfy the relation,

$$\frac{dw}{dx} = \Psi (x) + \beta (x), \qquad \ldots (9)$$

$\frac{dw}{dx}$ being the total slope of the centreline and $\Psi(x)$ is the rotation of the line elements due to bending only.

In this case we get the two field equations as,

$$\frac{d}{dx}\left[EI\frac{d\Psi}{dx}\right] + KGA\left(\frac{dw}{dx} - \Psi\right) = 0 \qquad \ldots (10)$$

and $$\frac{d}{dx}\left[KGA\left(\frac{dw}{dx} - \Psi\right)\right] + q = 0 \qquad \ldots (11)$$

where G is the shear modulus, K is the shear coefficient and A is the area of cross section.

Differentiating (10) with respect to x and using (11) we get the single field equation as

$$\frac{d^2}{dx^2}\left[EI\frac{d\Psi}{dx}\right] = q. \qquad \ldots (12)$$

In terms of dimensionless variable $X = \frac{x}{a}$ the field equation (12) when (2) and (3) are taken into account, becomes ,

$$\frac{E_o I}{a^3} \frac{d^2}{dX^2} \left[e^{-mX} \frac{d\Psi}{dx} \right] = q_o X^n \qquad \dots (13)$$

and the boundary conditions become

$$w = 0$$

and $\frac{d\Psi}{dX} = 0$ at $X = 1$ and $X = 2$. ... (14)

Following the same method as discussed in the previous section the solution of (13) satisfying (14) is obtained as,

$$w = a\,\Psi_o (X-1) + A_1 f(X) + B_1 g(X) + C_1 \theta(X) - D_1 \; \Phi(X) \quad \dots (15)$$

where $\phi(X) = (n+2) \int_1^X t^{n+1} e^{mt} \, dt + (1-2^{n+2}) \int_1^X e^{mt} \, dt$

and $D_1 \qquad A_1 \frac{12+11\nu}{60} (\frac{h}{a})^2$.

In deriving (15) we have replaced G by $E/2(1+\nu)$ and A by bh. We have also introduced $I = \frac{1}{12} bh^3$, where b and h are the dimensions of the rectangular cross section of the beam and we have chosen the value of shear co-efficient K for a rectangular cross section from a paper by Cowper [5] as,

$$K = \frac{10(1+\nu)}{12+11\nu}$$

ν being the constant Poisson's ratio.

The quantity Ψ_o appearing in equation (15) is the value of Ψ at $X=1$ and it satisfies the equation,

$$a\Psi_o + A_1 f(2) + B_1 g(2) + C_1\theta(2) - D_1\,\phi(2) = 0. \qquad \ldots (16)$$

Comparing equation (15) with equation (7) we see that the effect of shear deformation on the deflection in the nonhomogeneous beam lies in the last term in equation (15).

NUMERICAL RESULTS

To study the effect of nonhomogeneity as stipulated in (2) and the effect of nonuniform loading of the type (3) we have computed deflections for different values of m and n on taking Poisson's ratio $\nu = 1/3$. Deflection valuee have been computed at $X = 1.1\ (.1)2$ and the values of $R=\frac{E_0 I}{q_0 a}4w$ for technical beam theory as well as for Timoshenko beam theory have been displayed in Tables 1 to 4 to consider different cases of interest. Moreover, to study the effects of shear deformation for increasing values of h/a the deflections for technical as well as shear theory have been shown for a particular position of the beam in Table 5.

Table-1

Values of R for m = 0,n = 0 and h/a = .1

X =	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
R_{Tech}	.004087	.007733	.010587	.012400	.013020	.012400	.010587	.007733	.004087	0
R_{Timo}	.004205	.007942	.010862	.012713	.013347	.012713	.010862	.007942	.004205	0

Table - 2

Values of R for m = 1,h/a =.1 and different n

	X =	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
n = 0											
R_{Tech}		.017017	.032687	.045709	.054894	.059236	.058008	.050881	.038066	.020486	0
R_{Timo}		.017453	.033510	.046860	.056295	.060791	.059601	.052369	.039277	.021216	0
n = .5	X =	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
R_{Tech}		.020659	.039745	.055704	.067078	.072600	.071316	.062743	.047069	.025387	0
R_{Timo}		.021173	.040726	.057090	.068784	.074313	.073295	.064611	.048604	.026319	0
n = 1	X =	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
R_{Tech}		.025221	.048592	.068250	.082402	.089445	.088131	.077772	.058503	.031622	0
R_{Timo}		.025831	.049769	.069929	.084488	.091809	.090603	.080128	.060460	.032823	0
n = 1.5	X =	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
R_{Tech}		.030958	.059729	.084063	.101748	.110758	.109459	.096883	.073081	.039587	0
R_{Timo}		.031688	.061149	.086108	.104314	.113694	.112559	.099867	.075586	.041140	0
n = 2	X =	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
R_{Tech}		.038204	.073803	.104071	.126267	.137825	.136609	.121273	.091732	.049799	0
R_{Timo}		.039083	.075528	.106574	.129436	.141486	.140512	.125069	.094950	.051816	0

Table - 3

Values of R for n = 1, h/a = .1 and different m

	X =	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
m = 0											
R_{Tech}		.006004	.011403	.015691	.018485	.019531	.018714	.016071	.011797	.006258	0
R_{Timo}		.006165	.011695	.016084	.018945	.020021	.019195	.016501	.012131	.006450	0
	X =	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
m = .5											
R_{Tech}		.012245	.023431	.032588	.038877	.041641	.040461	.035216	.026161	.014004	0
R_{Timo}		.012551	.024006	.033386	.03984	.042701	.041535	.036209	.026960	.014478	0
m = 1	X =	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
R_{Tech}		.025221	.048592	.068250	.082402	.089445	.088131	.077772	.058503	.031622	0
R_{Timo}		.025831	.049769	.069929	.084488	.091809	.090603	.080128	.060460	.032823	0
	X =	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
m = 1.5											
R_{Tech}		.052475	.101717	.144163	.176031	.193558	.193371	.173043	.131880	.072029	0
R_{Timo}		.053749	.104229	.147828	.180697	.198983	.199197	.178756	.136767	.075121	0
	X =	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
m = 2											
R_{Tech}		.110298	.214942	.307145	.379021	.421948	.427315	.387815	.299568	.165430	0
R_{Timo}		.113087	.220521	.315423	.389760	.434701	.441328	.401904	.311949	.173491	0

Table - 4

Value of R for m = 1, n = 1 and different h/a

X = h/a=.1	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
R_{Timo}	.025831	.049769	.069929	.084488	.091809	.090603	.080128	.060460	.032823	0

X = h/a=.01	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
R_{Timo}	.025227	.048604	.068267	.082423	.089486	.088156	.077795	.058523	.031634	0

Table - 5

Values of R for m = 1, n = 0, X = 1.5 and different h/a

$R_{Tech} = 0.059236$

h/a	R_{Timo}
0.1	0.060791
0.2	0.065458
0.3	0.073236
0.4	0.084124
0.5	0.098123

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COMMON FIXED POINT THEOREMS IN METRIC LINEAR AND UNIFORM SPACES

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In this paper we generalize and improve a fixed point theorem in normed linear spaces due to Khan, Imdad and Sessa [3] to matric linear spaces and also prove a common fixed point theorem in uniform spaces.

Let (X,d) be a metric linear space. It has ball conveity if $r \in R^+$, x, $y \in X$ with $d(x,0) = d(y,0) = r \Longrightarrow d(\frac{x+y}{2}, 0) \leq r$.

In was observed by Sastry and Naidu [6] that in a metric linear space (X,d) with ball convexity (B.C.), the function d(tx, 0) is a increasing function to t on R^+ for any x in X. We first observe the following:

LEMMA 1. Suppose (X, d) is a metric linear space with ball convesity, $\{a_n\}$ is a sequence in X and $\{\alpha_n\} \subseteq [0, 1]$ such that $\{\alpha_n\}$ is bounded away from zero. If $\{\alpha_n a_n\}$ converges to zero then $\{a_n\}$ also converges to zero.

Proof. Since $\{\alpha_n\}$ is bounded away from zero, there exists a positive integer N such that $\alpha_n > \frac{1}{N}$ for sufficiently large n. Since d(tx, 0) is an increasing function of t or R^+ for any x in X, we have $d(a_n, 0) = d(\frac{1}{\alpha_n}\alpha_n a_n, 0) \leq d(N\alpha_n a_n, 0)$, for sufficiently large n,

$\leq N d(\alpha_n a_n, 0)$ since d is translation invariant.

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Since $\{\alpha_n a_n\}$ converges to zero it follows that $\{a_n\}$ also converges to zero.

Definition (Jungck [2]). Two self-maps f and g on a metric space (X,d) are said to be compatible on X iff $\operatorname*{Lt}_{n\to} d(fgx_n, gfx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\operatorname*{Lt}_{n\to} fx_n = u = \operatorname*{Lt}_{n\to} gx_n$ for some u in X.

THEOREM 2. Let (X,d) be a metric linear space with ball convexity and K be a nonempty convex subset of X; f, g, h be self-mappings of K satisfying

$$(2.1)\ d(fx,gy) \leq \phi(d(hx,hy), d(hx,fx)+d(hy,gy), d(hx,gy) + d(hy,fx), d(hx,fx)+d(hx,gy), d(hy,gy)+d(hy,fx))$$

for all x, y in K where $\phi : R^{+5} \to R^{+}$ is an upper semi continuous function such that (i) $\phi(0, t, t, t, t) < t$ for $t > 0$, (ii) $\phi(t, 0, 2t, t, t) < t$ for $t > 0$.

Suppose there exists a sequence $\{C_n\}$ [0,1] which is bouded away from zero.

Suppose for $x_o \in K$, there exists a sequence $\{x_n\}$ such that $hx_{2n+1} = (1-C_n)\, hx_{2n} + C_n\, fx_{2n}$, $hx_{2n+2} = (1 - C_n)\, hx_{2n+1} + C_n\, gx_{2n+1}$ for $n \geq 0$. Suppose either f and h or g and h are compatible on K. If $\{hx_n\}$ converges to z in K and h is continuous at z then z is the unique fixed point of f, g and h.

Proof:- Clearly $hx_{2n+1} - hx_{2n} = C_n\,(fx_{2n} - hx_{2n})$.

By Lemma 1, $fx_{2n} - hx_{2n} \to 0$ as $n \to \infty$ and hence $fx_{2n} \to z$ as $n \to \infty$. Similarly $gx_{2n+1} \to z$ as $n \to \infty$.

Suppose f and h are compatible on K. then

$$d(fhx_{2n}, hfx_{2n}) \to 0 \text{ as } n \to \infty .$$

Since h is continuous at z we have $hfx_{2n} \to hz$, $h^2x_{2n} \to hz$ as $n \to \infty$ and hence $fhx_{2n} \to hz$ as $n \to \infty$. Now,

$$\begin{aligned} d(fhx_{2n}, gx_{2n+1}) \leq \emptyset\ (&d(h^2x_{2n}, hx_{2n+1}), d(h^2x_{2n}, fhx_{2n}) \\ &+ d(hx_{2n+1}, gx_{2n+1}), d(h^2x_{2n}, gx_{2n+1}) \\ &+ d(hx_{2n+1}, fhx_{2n}), d(h^2x_{2n}, fhx_{2n}) \\ &+ d(h^2x_{2n}, gx_{2n+1}), d(hx_{2n+1}, gx_{2n+1}) \\ &+ d(hx_{2n+1}, fhx_{2n})). \end{aligned}$$

Letting $n \to \infty$, we get

$d(hz, z) \leq \emptyset\ (d(hz, z), 0, 2\ d(hz, z)\ , d(hz, z))$.

By (ii), we get $hz = z$.

Suppose g and h are compatible on K.

Then putting $x = x_{2n}$, $y = hx_{2n+1}$ in (2.1) and letting $n \to \infty$ and using (ii) one can prove that $hz = z$.

Now,

$$\begin{aligned} (2,2)\ d(fz, gx_{2n+1}) \leq \emptyset\ (&d(hz, hz_{2n+1}), d(hz, fz) + d(hx_{2n+1}, \\ &gx_{2n+1}), d(hz, gx_{2n+1}) + d(hx_{2n+1}, fz), \\ &d(hz, fz) \\ &+ d(hz, gx_{2n+1}), d(hx_{2n+1}, gx_{2n+1}) \\ &+ d(hx_{2n+1}, fz)). \end{aligned}$$

Letting $n \to \infty$, we get

$d(fz, z) \leq \emptyset\ (0, d(z, fz), d(z, fz), d(z, fz)\ d(z, fz))$.

By (i), we get $fz = z$.

Similarly putting $x = x_{2n}$, $y = z$ in (2.1) and letting $n \to \infty$ and using (i), one can prove that $gz = z$.

Thus z is a common fixed point of f, g and h. uniqueness of common fixed point follows easily from (2.1) and (ii).

COROLLARY 3. Theorem 2 holds if the inequality (2.1) is replaced by

(3.1) $d(fx, gy) \leq a\, d(hx, hy) + b\, [d(hx, fx) + d(hy, gy)]$

$+ c\, [d(hx, gy) + d(hy, fx)]$

for all $x, y \in K$ where $a + b + c \geq 0$, $b + c < 1$, $a + 2c < 1$

Proof . Corollary 3 follows by selecting $\emptyset\, (t_1, t_2, t_3, t_4, t_5) = at_1 + bt_2 + ct_3$ in Theorem 2.

REMARK 4 . In Corollary 3, the assumption '$\{hx_n\}$ converge to z in K' cannot be dropped even when $g = f$, h = identif map, $C_n = 1$ for all n in view of the following example.

Example. Define $F ; R_+ \to R_+$ as $F(t) = \begin{cases} t, & 0 \leq t \leq 1 \\ 1, & t \geq 1 \end{cases}$

Then (R, d), where $d(x, y) = F(|x-y|)$ for all $x, y \in R$, is Complete matric linear space with ball convexity.

Let $K = [0, \infty)$. Define $f : K \to K$ as $fx = x+1$. Then $d(fx, fy) \leq \frac{1}{2}\, d(x,y) + \frac{1}{4}\, [d(x, fx) + d(y, fy)]$ for all $x, y \in K$.

Here $a = \frac{1}{2}$, $b = \frac{1}{4}$, $c = 0$.

For any $x_0 \in K$, the sequency $\{x_n\}$ defined by $x_{n+1} = fx_n$, $n = 0, 1, 2, \ldots$ does not converge to any $z \in K$.

f has no fixed point.

REMARKS. The following example shows the situation where Corollary 3 is applicable and Theorem 7 is not applicable even when $g = f$, $C_n = 1$ for all n.

Example. Let $X = K = (0, \infty)$, $d(x, y) = |x - y|$ for all $x, y \in K$. Let $fx = x^2$, $hx = 2x^2$ for $x \in K$. Clearly $f(X) = h(X)$ and $d(fx, fy) = \frac{1}{2}\, d(dx, hy)$ for all $x, y \in K$.

Clearly f and h are not weakly commuting. But f and h are compati

For $x_o = \frac{1}{2}$ there exist $x_1 = \frac{1}{2\sqrt{2}}$, $x_2 = \frac{1}{2^2}$, $x_3 = \frac{1}{2^2\sqrt{2}}$,..... such that $hx_{n+1} = fx_n = \frac{1}{2^{n+2}} \to 0 \in K$.

Clearly h is not linear. Zero is the only common fixed point of f and h.

REMARKS 6. In Corollary 3, the assumption "h is continuous at $z \in K$" cannot be dropped even when $g = f$, $C_n = 1$ for all n in view of the following example.

Example . Let $X = [0,1] = K$, $d(x,y) = |x-y|$ for all $x,y \in K$. Define $fx = \frac{x}{2}$ for all $x \in K$;

$hx = x$ if $x \neq 0$, $h(0) = 1$. Here $f(X) \not\subset h(X)$, h is not linear.

Clearly $d(fx, fy) \leq \frac{1}{2} d(hx, hy) + \frac{1}{2} [d(hx,fx) + d(hy,fy)]$ for all x, $y \in K$. f and h are weakly commuting and hence compatible. For $x_o \neq 0$ there exists $\{x_n\}$ such that $hx_{n+1} = fx_n \to 0 \in K$. Clearly h is not continuous at 0. But 0 is not the common fixed point of f and h.

Corollary 3 is an improvement of the following theorem of [3].

THEOREM 7 ([3]). Let f and h be two self-mappings of a linear normed space X such that f, h is weakly commuting pair, h is continuous, linear and $f(X) \subseteq h(X)$. Further, we have

$$||fx - fy|| \leq a\,||hx - hy|| + b[\,||hx - fx|| + ||hy - fy||\,]$$
$$+ c\,[\,||hx - fy|| + ||hy - fx||\,]$$

for all $x, y \in X$ where $a + b + c \geq 0$, $a + 2b + 2c \leq 1$. Let $x_o \in X$ and $\{C_n\}$ be a sequence satisfying $0 < C_n \leq 1$ for $n \geq 0$ and $\underset{n\to\infty}{\text{Lt}} C_n > 0$. Suppose the sequence $\{hx_n\}$ defined by $hx_{n+1} = (1 - C_n)\, hx_n + C_n\, fx_n$ (for $n \geq 0$) converges to a point z in X. Then z is a coincidence point of f and h. If $b > 0$ then fz is the unique common fixed of f and h.

In the proof of Theorem 7, they [3] used $a \geq 0$, $b \geq 0$ without mentioning it in the hypothesis for the existence of coincidence point.

REMARK 8. Theorem 2 is true if either f and h or g and h are compatible and h is continuous at z are replaced by either $fh = hf$ or $gh = hg$ and h^k is continuous at z for some positive integer k.

One can prove the theorem in Remark 8 by the similar lines of cases (7) and (2) in Theorem (1.1 of Naidu and Prasad [5].

Now we prove a theorem in uniform spaces.

Let (X, U) be a Hausdorff sequentially complete uniform space and let P be a fixed family of pseudo-metrics on X which generates the uniformity U. For any pseudo-metric p on X and $r > 0$, let $V_{(p,r)} = \{(x,y,) \mid x,y, \in X, p(x,y) < r\}$.

Let $G = \{V_n \mid V_n = \bigcap_{i=1}^{n} V_{(pi,\ ri)}, r_i > 0, p_i \in P, i = 1, 2, \ldots, n\}$.

Two self-maps f and h in (X,U) said to be compatible if $p(fhx_n, hfx_n) \to 0$ as $n \to \infty$ whenever $\{x_n\}$ is a sequence such that $\operatorname{Lt}_{n\to\infty} fx_n = u = \operatorname{Lt}_{n\to\infty} hx_n$ for some u in X where p is any pseudo-metric on X.

THEOREM 9. Let f and h be self-maps of X such that h is continuyous, $f(X) \subseteq h(X)$ and f and h are compatible on X. Also for any V_1, V_2, V_3, V_4, V_5 in G and x, y in X, $(hx, hy) \in V_1$, $(hx, fx) \in V_2$, $(hy, fy) \in V_3$, $(hx, fy) \in V_4$, $(hy, fx) \in V_5$ implies

$$(9.1)\quad (fx, fy) \in a_1(x, y)\, V_1 \circ a_2(x,y)\, V_2 \circ a_3(x,y)\, V_3 \circ a_4(x,y)\, V_4 \circ a_5(x, y)\, V_5$$

where $a_i : X \times X \rightarrow R^+$ for i = 1, 2, 3, 4, 5 such that

$$\sup_{x,y \in X} \sum_{i=1}^{5} a_i (x,y)] = \lambda < 1.$$

Then f and g have a unique common fixed point z in X.

Proof. Let x, y be arbitrary in X and V be arbitrary in G. Let p be the Minkowski pseudo-metric of V and put $p(hx,hy) = r_1$, $p(hx,fx) = r_2$, $p(hy,fy) = r_3$, $p(hx,fy) = r_4$, $p(hy,fx)=r_5$. Then with arbitary > 0, we have

$(hx,hy) \in (r_1 + \varepsilon) V$, $(hx,fx) \in (r_2+\varepsilon) V$, $(hy,fy) \in (r_3+\varepsilon) V$, $(hx,fy) \in (r_4+ \varepsilon) V$, $(hy,fx) \in (r_5+\varepsilon) V$.

Write $a_i(x, y) = a_i$ for the sake of convenience.

Therefore by (9.1), we have

$(fx,fy) \in a_1(r_1+\varepsilon)V_o\, a_2(r_2+\varepsilon)V_o\, a_3(r_3+\varepsilon)V_o\, a_4(r_4+\varepsilon)V_o\, a_5(r_5+\varepsilon)V$

Since ε is arbitrary we have

$p(fx, fy) \leq a_1\, p(hx,hy) + a_2\, p(hx,fx) + a_3\, p(hy,fy) + a_4 p(hx, fy) + a_5 p(hy,fx)$ and it follows that

(9.2) $p(fx,fy) \leq \lambda \max \{p(hx,hy), p(hx,fx), p(hy,fy),$

$p(hx,fy), p(hy,fx)\}$ for all x, y in X.

For x_o in X, define $\{x_n\}$ such that $fx_n = hx_{n+1}$ for n=0,1,2,... It can be possible since $f(X) \subseteq h(X)$.

One can prove, by following the procedure as in Das and Naik [1] or Naidu and Prasad [4], that $\{fx_n\}$ is Cauchy. Hence $\{fx_n\}$ converges to some $z \in X$. By continuity of h and compatibility of f and h we get $fhx_n \rightarrow fx$ as $n \rightarrow \infty$.

Using (9.2) to the terms $p(fhx_n, fz)$ and $p(fhx_n, fx_n)$ and letting $n \rightarrow \infty$ we get $p(hz,fz) = 0 = p(hz, z)$.

So (hz, fz) and (hz, z) are in V for all V in G. Hence hz = fz = z. Uniqueness of common fixed point follows easily by (9.2).

REMARK 10. Theorem 9 is true if f and h are compatible and h is continuous are replaced by fh = hf and h^k is continuous for some positive integer k.

Remark 10 is an improvement of Theorems 2.1, 3.1, 3.2, 4.2 of [1].

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STUDIES ON SOIL CONSERVATION VALUE OF COMMON PLANTS OF SHIVALIKS

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ABSTRACT

In the present study soil conservation value of *Cynodon dactylon*, *Adhatoda vasica*, *Ipomea carnea* *Cymbopogon martimii* and *Sida sps* has been investigated and it has been found that *Cynodon dactylon*, *Adhatoda vasica* and *Cymbopogon martimii* are the suitable species for plantation on hilly slopes because thier soil conservation value is maximum, being 82.5%, 72.1% and 48.4%, respectively. The moisture content of soil which helps in formation of soil texture under these plant species was also higher being 18.5%, 20.5% and 15.2%, respectively, (Fig.1).

Key words : Soil conservation value, soil moisture, leaf area index.

INTRODUCTION

Danger is looming large in Shivalik, known as a delicate (Kuchcha) mountain range. According to the botanists the 1000 feet high mountain surrounding the famous Mansa Devi temple faces collapse in a few years as a result of induscriminate cutting of plants, pilgrims activties and uncontrolled grazing. People felled trees at Bilkeshwar too build a Bilkeshwar colony. Not only did they cut valuable tree but even diverted a stream into the Ganga making themselves susceptible to land slide and floods. These hills are surrounded by many villages. The grazing animals as well as wood and fodder collectors of these villages reduce the forest wealth. It leads to loss

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of organic layer and humus which is helpful in conserving soil moisture. It is estimated that approximately 40 crore animals grage in Indian forests. These result in soil erosion on hilly slopes in many parts of India converting them into barrenlands. Shivaliks and Aravali hills of India are facing severe ill effects of uncontrolled grazing by heeds of cattle, Maheshwari [6]. Soil erosion from Shivalik in Hardwar and river bank in downstream was found to the enormous during rains. Ambasht [1] has found the soil conservation values of *Euphorbia munja* (92-96%), *Cynodon dactylon* (90-97%) and Weed *Euphorbia hirta* (10-12%). Besides, several grasses have been found suitable for checking soil erosion in the Nilgiri hill e.g. *Eragrostis amabitis*, *E. cervula*, *Cynodon dactylon* and *Dactylis glomerata*,, Bhatia [3]. Hence, to check soil erosion, the most effective and natural process would be to plant suitable species at suitable sites, Shanker [7].

MATERIALS AND METHODS

The selection of plants has been done on the basis of their economic importance, root habit and ground cover. Plant were planted on the experimental sloping plots each of 1x3m in size (*In the G.Kangri botanical garden*) with an inclination angle of 13°. At the base of the slope cemented runoff collectors were constructed to receive the water runoff and eroded soil from the plots. Water was sprayed upon the plots from a height of 1mt for 30 min at a constant speed of 16l/min showering done at 15 days intervals, Ambasht [2].

(a) $$CV = 100-\left(\frac{S_{wp}}{S_{wo}} \times 100\right)$$

where - CV = Conservation value

S_{wp} = Weight of soil washed from vegetated plots

S_{wo} = Weight of soil eroded from the bare plot

(b) LAI = TLA/TGA

where
LAI = Leaf area index
TLA = Total leaf area
TGA = Total ground area

RESULTS AND DISCUSSION

In the present study soil conservation value and soil moisture content of *Adhatoda vasica, Ipomea carnea, dactylon, Cymbopogon martimii & Sida sps has* been investigated. As evident from table (I) *Cynodon dact-ylon* and *Adhatoda vasica* showed higher conservation value i.e. 49.2% and 45.0% the moisture content of soil, which helps in formation of soil texture under these plants was also higher being 19.6% and 21.9%, respectively.

Ipomea carnea, Sida sps and *Cymbopogon martimii* showed lower conservation value standing at 25.8%, 37.8%, 37.6% and 39.3%, respectively. On day 15 the conservation value of *Cynodon dact-ylon* and *Adhatoda vasica* was found to be 68.9% and 52.6%, respectively and the moisture content of soil stood at 16.6% and 20.2%, respectively. *Ipomea carnea , Sida sps* and *Cymbopogon martimii* showed lower conservation value standing at 28.7%, 39.2% and 42.8%, while the moisture content of soil was also lower i.e. 15.4%, 15.8% and 13.7%, respectively. On day 30 also the maximum soil conservation value was recorded in the case of *Cynodon dactylon* and *Adhatoda vasica* standing at 75.5% and 68.0%, the moisture content of soil was also higher being 20.7% and 21.4%, respectively

The experiment was conducted for a period of 45 days. Best results in respect of soil conservation value and soil moisture content were obtained with *Cynodon dactylon* and *Adhatoda vasica* which on day 45 stood respectively at

82.5%, 18.5% and 72.1% and 20.2%. *Ipomea carnea*, *Sida sps* and *Cymbopogon martimii* showed soil conservation value and soil moisture content which on day 45 stood at 36.2%, 14.6%, 12.2% and 48.4%, 15.0% respectively. The result shows that relatively high soil conservation value and soil moisture constent as obtained under the effect of *Cynodon dact ylon* and *Adhatoda vasica* are attributable to better higher leaf area index of these plants i.e. 0.89 and 0.98 cm^2, respectively. Besides, thir root system also was better developed which could tightly bind the soil particles against erosion. The higher leaf area of these plants provided a cover to the soil against the impact of rain drops so that the soil particles were relatively mildly hit resulting in the checking of soil erosion. The present findings are supported by earlier work where it was found that the dense grass cover checks the velocity of the rain drops and result is more water entering the soil and no erosion because their canopy and profuse root system tightly bind the soil against erosion, Donahue [4]. Grasses can be grown on lands which are not suitable for cultivation of crops. They are also used in protecting bunds, water ways, badly eroded areas etc. Kanitkar et al.[5].

In the light of the present study *Cynodon dact-ylon* and *Adhatoda vasica* are recommended for growing in the Shivalik for soil and moisture conservation. *Adhatoda vasica* is an important medicinal plant and hence its plantation in the Shivalik shall add to the medicinal plant wealth of the region that has been dwindling over the years as a consequence of over exploitation and ecodegradation in the region.

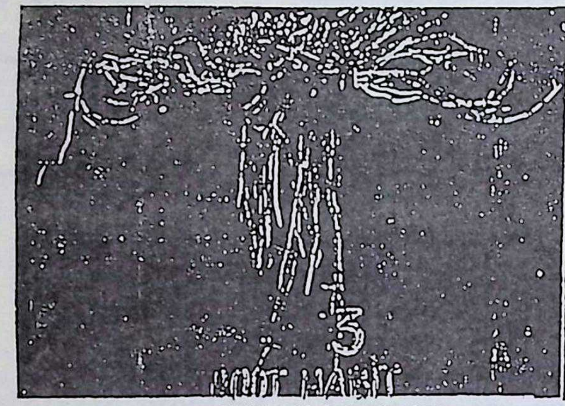
TABLE-1

MOISTURE CONTENT AND SOIL CONSERVATION VALUE OF DIFFERENT PLANTS

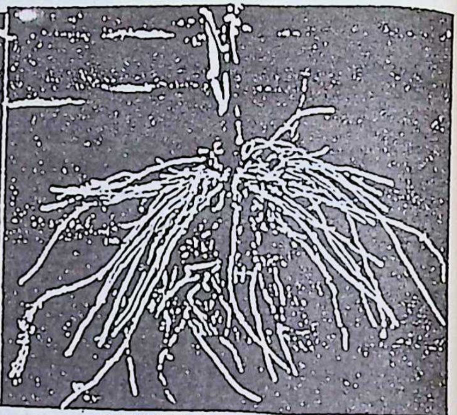
INTERVAL DAY	NAME OF PLANT SPS.	AIR T °C	SOIL T °C	HEIGHT OF PLANT (Cm)	LAI (Cm^2) (mean)	SOIL MOIS-TURE %	SCV *%	% OF NITROGEN
0	A	24.0	22	-	-	11.1	NIL	0.010
	B	24.0	21	12.6	0.06	21.9	45.0	0.012
	C	24.0	22	14.3	0.031	18.3	25.8	0.012
	D	24.0	21.5	13.2	0.52	19.6	49.2	0.013
	E	24.0	18	14.0	0.026	14.9	39.3	0.013
	F	24.0	16.5	10.8	0.05	16.2	37.6	0.012
15	A	24.8	23.6	-	-	10.8	NIL	0.010
	B	24.8	21.8	18.7	0.49	20.2	52.6	0.012
	C	24.8	23.0	16.8	0.08	15.4	28.7	0.011
	D	24.8	21.6	16.3	0.63	16.6	68.9	0.013
	E	24.8	19.6	18.9	0.032	13.7	42.8	0.013
	F	24.8	16.8	12.4	0.09	15.8	39.2	0.011
30	A	28.6	26.0	-	-	8.7	NIL	0.008
	B	28.6	24.2	29.8	0.86	21.4	68.0	0.012
	C	28.6	27.8	38.2	0.38	16.8	32.7	0.011
	D	28.6	24.0	27.5	0.72	20.7	76.5	0.013
	E	28.6	26.7	29.2	0.036	15.2	46.6	0.014
	F	28.6	25.4	26.8	0.42	14.4	41.2	0.012
45	A	29.7	26.9	-	-	6.0	NIL	0.006
	B	29.7	24.0	34.8	0.98	20.2	72.1	0.013
	C	29.7	29.8	39.8	0.66	14.6	36.2	0.012
	D	29.7	24.5	28.4	0.89	18.5	82.5	0.013
	E	29.7	26.4	30.8	0.47	15.0	48.4	0.014
	F	29.7	26.0	28.8	0.56	12.2	45.6	0.013

A = Control. B = *Adhatoda vasica* C = *Ipomea carnea* D = *Cynodon dactylon*.
E = *Cymobopogon martimii* F = *Sida sps.*

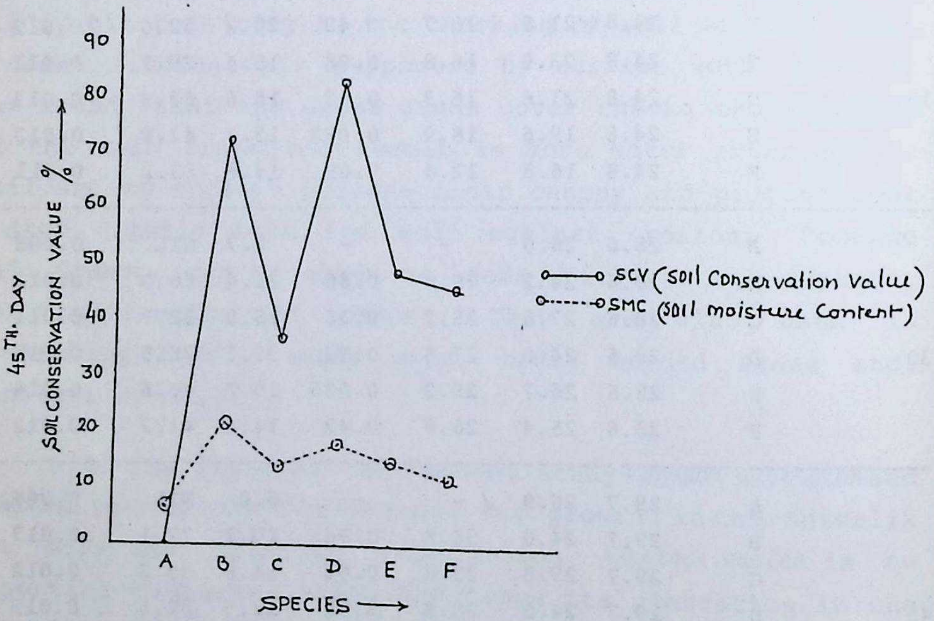
* Soil Conservation Value.



CYNODON ROOT HABIT



ADHATODA ROOT HABIT



(Fig 1)

A= control B= Adhatoda vasica, C = Ipomea carnea, D= Cynodon dactylon
E = Cymbopogon martinii F= Sida sps.

Fig 1: Graphical representation of SCV and SMC by different plant species

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SIGNIFICANCE OF THE CONSTANT TERM OF A HERMITE POLYNOMIAL OF EVEN DEGREE

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ABSTRACT

We present new proofs of the definite parity of the Hermite polynomials and the recurrence relations satisfied by them. We also show that *all* the Hermite polynomials will be simple polynomials if the constant term of *any* Hermite polynomial of even degree is non-zero.

Keywords and Phrases : Hermite polymonials, Wronskian, Parity, Recurrence relations, constant term of a Hermite polynomial of even degree.

Mathematics Subject Classification Number:33A65

INTRODUCTION

The purpose of this paper is to present new proofs of the definite parity of the Hermite polynomials [1,2,4] and their recurrence relations and also to look at the significance of the constant term of *any* Hermite polynomial of even degree. If $H_{2n}(0)$ is non-zero for a *single* positive integer $n \geq 0$, the entire set of the Hermite polynomials, $[H_m(x)]$, will be a simple set of polynomials, each polynomial in the set having a non-zero leading term and hence a precise

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degree [4]. The differential equation, recurrence relations, and zeros of the Hermite polynomials are the windows through which we look at the beauty of $H_{2n}(0)$, the constant term of $H_{2n}(x)$.

WRONSKIAN, PARITY AND RECURRENCE RELATIONS

The basis for our analysis is the differential equation satisfied by the Hermite polynomials [1,2,4]:

$$(D^2 - 2\,x\,D + 2p)\,H_p(x) = 0,\ p = 0,1,2,\ldots, \qquad (1)$$

where

$$D \equiv d/dx. \qquad (2)$$

The Wronskian of the two solutions Y_1 and Y_2 of the Hermite differential equation (HDE), $W_{HDE}(x)$, obtained using Abel's formula [3,5,6], is given by

$$W_{HDE}(x) = Y_1 Y_2{}' - Y_2 Y_1{}' = W_{HDE}(0)\exp(x^2). \qquad (3)$$

Since $W_{HDE}(x)$ is an *even* function of x and also an *infinite* series, the two linearly independent solutions of the HDE cannot be polynomials simultaneously. (The Wronskian of two polynomial solutions will be another polynomial, leading to $W_{HDE}(0) = 0$.) Moreover, the two linearly independent solutions of the HDE cannot have the same parity. (The Wronskian will be an odd function in this case, yielding $W_{HDE}(0) = 0$.) As (1) is a homogeneous second order linear differential equation, it has $H_p(x) \equiv 0$ as a trivial solution. But we are interested in finding a non-trivial polynomial solution satisfying (1). When $x \rightarrow -x$, $D \rightarrow -D$, $D^2 \rightarrow D^2$, and $-2xD \rightarrow -2xD$. Hence if $H_p(x)$ is a non-trivial polynomial solution of the HDE, $H_p(-x)$ is a also solution of the same HDE. Since both $H_p(x)$ and $H_p(-x)$ are non-trivial polynomials, they are simply proportional to each other and hence $H_p(x)$ has a definite parity $(-1)^p$:

$$H_p(-x) = (-1)^p H_p(x). \tag{4}$$

We next obtain, from the HDE, the differential recurrence relation satisfied by the Hermite polynomials. From (1), it is clear that $D(D^2-2xD+2p)\ H_p(x) = D\ 0 = 0$. Hence we have.

$$[D^2 -2xD+2(p-1)]\ [DH_p(x)=0, p\geq 1. \tag{5}$$

Thus both $H_{p-1}(x)$ and $DH_p(x)$, $p \geq 1$, are solutions of the same HDE. Since both are non-trivial polynomials (they have the same parity too), they cannot be the linearly independent solutions of the HDE. Hence $DH_p(x) = C_p\ H_{p-1}(x)$, $p \geq 1$, C_p being the constant of proportionality. The familiar choice is $C_p = 2p$, for all $p \geq 1$. (The leading coefficient of *any* $H_p(x)$ cannot be obtained from the HDE. The above choice makes the leading coefficient of $H_p(x)$ to be 2^p.) Thus the differential recurrence relation is [1,2,4]:

$$DH_p(x) = 2p\ H_{p-1}(x), \quad p \geq 1. \tag{6}$$

From (1), we have $[D^2 - 2\ x\ D + 2(N+1)]\ H_{N+1}(x)=0$. Eliminating the derivatives using (6), we arrive at the three-term recurrence relation satisfied by the Hermite polynomials [1,2,4] :

$$H_{N+1}(x) = 2\,x\,H_N(x) - 2\,N\,H_{N-1}(x),\ N \geq 1. \tag{7}$$

Using (4), we have $H_{2m+1}(0) = 0$. One may be tempted to ask whether $H_{2m}(0)$ can also follow suit. We show that the answer is definitely in the negative.

SIGNIFICANCE OF $H_{2m}(0) \neq 0$ AS SEEN THROUGH THE WINDOW OF THE HDE

The HDE for even degree polynomials is

$$(D^2 - 2\,x\,D + 4m)\,H_{2m}(x) = 0,\ m \geq 0. \tag{8}$$

Since $H_{2m}(x)$ is an even polynomial in x (see (4)), its derivative with respect to x is an odd polynomial in x and hence vanishes at x = 0. If the constant term of $H_{2m}(x)$ were zero, then the solution of (8) and its derivative would be zero at the same point x = 0. As an initial value problem in the theory of second order linear differential equations has a unique solution [5,6], the solution of (8) would be identically zero. As we are interested in having a non-trivial polynomial solution of the HDE, we must choose $H_{2m}(0)$ to be non-zero.

What we give below is a "constructive" proof of the fact that $H_{2m}(x)$ is non-trivial, with a non-zero leading term. This is an alternative to the one generally given in books [1,6] for finding the polynomial solution from the HDE. Differentiating both sides of (8) with respect to x successively 2k times, using the Leibnitz theorem, we have

$$D^{2k+2}H_{2m}(x)\big|_{x=0} = 4(k - m)\,D^{2k}\,H_{2m}(x)\big|_{x=0}. \tag{9}$$

As $H_{2m}(0)$ is non-zero, $m \geq 1$, iteration of (9) shows that $D^{2s} H_{2m}(x)$ has to be non-zero at $x = 0$ when $0 \leq s \leq m$. If

$$H_{2m}(x) = \sum_{s=0}^{m} A_{2s} x^{2s}, \tag{10}$$

then

$$A_{2s} = (-1)^s 4^s m! H_{2m}(0)/[(2s)! (m-s)!] \neq 0. \tag{11}$$

Using (1), (4) and (6), we now show that $H_{2m}(0) \neq 0$ will also make $H_{2m+1}(x)$ a non-trivial polynomial. The Hermite polynomial of odd degree is an odd polynomial and hence is zero at the origin. The differential equation satisfied by it implies that it is not identically zero since its derivative is different from zero at the origin: $DH_{2m+1}(x)|_{x=0} = 2(2m+1)H_{2m}(0) \neq 0$.

It follows from the differential recurrence relation and (8) that

$$\begin{aligned} 4(2m+2)(2m+1) H_{2m}(0) &= D^2 H_{2m+2}(x)|_{x=0} \\ &= -4(m+1) H_{2m+2}(0). \end{aligned} \tag{12}$$

Hence $H_{2m+2}(0) \neq 0$, if $H_{2m}(0)$ is non-zero for a particular m. Iteration of (12) now yields that $H_{2m}(0) \neq 0$, for all $m \geq 0$. Since every non-zero $H_{2m}(0)$ saves both $H_{2m}(x)$ and $H_{2m+1}(x)$ from being identically zero, it is clear that a single non-zero $H_{2m}(0)$ can save the entire set of Hermite polynomials.

THROUGH THE THREE-TERM RECURRENCE RELATION AND ZEROS

From the three-term recurrence relation, (7), we have

$$H_{2m}(0) = -2\,(2m-1)\,H_{2m-2}(0),\ m \geq 1\ . \qquad (13)$$

As every polynomial belonging to a set of simple polynomials has a precise degree with a non-zero leading term [4,11], $H_0(0)$ cannot be zero if we demand the Hermite polynomials to be simple polynomials. From (13) it follows that $H_{2m}(0) \neq 0$, $m \geq 0$.

If the constant term of $H_{2r}(x)$ were zero for some $r \geq 1$, then $H_{2r}(x) = x^2\,[A_{2r}\,x^{2r-2} + \ldots + A_2]$. (Here we use only the fact that $H_{2r}(x)$ has a definite parity (see (4)) and hence it contains only even powers of x.) Therefore, $H_{2r}(x)$ would have a zero of multiplicity 2 at x = 0, contradicting the fact that the zeros of real orthogonal polynomials are distinct [4,11]. (This result is a consequence of the orthogonal property [4,11] which itself is a natural consequence of the Sturm-Liouville form of the HDE [1,3,5]. See Section 6.) Hence $H_{2r}(0) \neq 0$, for all $r \geq 0$.

OTHER WINDOWS

Readers may wonder why we have not looked either through the window of the generating function [1,2,4]

$$\exp(2xt - t^2) = \sum_{m=0}^{\infty} H_m(x)\, t^m/m!\,, \qquad (14)$$

or through that of the Rodrigues' representation [1,2,4,

$$H_n(x) = (-1)^n \exp(x^2)\, D^n(\exp(-x^2)),\ n \geq 0, \qquad (15)$$

or through that of the simple but beautiful relation [1,7,8,9]

$$H_n(x) = (2x - D)^n\, 1,\quad n \geq 0. \qquad (16)$$

Each one of the above three relations has been proved to be a starting point for studying the properties of the Hermite polynomials. For proofs, see Refs. [2], [7], and [8,9], respectively. Like the power series expansion [1,2,4] of $H_n(x)$, they specify the Hermite polynomials completely, leaving no room for any speculation whether $H_{2n}(0)$, $n \geq 0$, can be zero or not.

STRUM-LIOUVILLE FORM AND SZEGÖ'S RESULT

The Strum-Liouville form of the HDE is [1,3,5]

$$[L + 2p \exp(-x^2)]\, H_p(x) = 0, \quad p \geq 0, \qquad (17)$$

where the Strum-Liouville operator L is $D \exp(-x^2) D$. As the eigenfunctions of the Sturm-Liouville operator belonging to distinct eigenvalues are orthogonal with respect to the weight function [1,3,5],

$$\int_{-\infty}^{+\infty} \exp(-x^2)\, H_m(x)\, H_n(x)\, dx = 0, \qquad m \neq n. \qquad (18)$$

The even nature of the associated weight function and the symmetry of the interval of orthogonality with respect to the origin demand, according to an elegent and general result from Szegö [11] that $H_n(x)$ *can contain* only those powers of x which are congruent to n(mod 2), i.e., it is an odd or even polynomial according to whether n is odd or even. Using Szegö's result [11], the properties of the zeros [4,11] of $H_n(x)$ and Descartes' rule of signs, it has been shown [10] that $H_n(x)$ *does contain* only those and all those powers of x which are congruent to n(mod 2). Hence in

$$H_{2n}(x) = \sum_{m=0}^{n} A_{2m} x^{2m}, \; n \geq 0, \tag{19}$$

$$H_{2n+1}(x) = x \sum_{s=0}^{n} A_{2s+1} x^{2s}, \; n \geq 0, \tag{20}$$

of the coefficients A_{2m} and A_{2s+1} can be zero. As simple corollaries, we have,

$$H_{2n+1}(0) = 0, \quad H_{2n}(0) \neq 0, \quad n \geq 0 \tag{21}$$

With $H_0(0) = 1$, it follows from (12) or (13) and iteration that

$$H_{2n}(0) = (-1)^n \, (2n)!/n! \neq 0, \quad n \geq 0. \tag{22}$$

CONCLUSIONS

What a tremendous potential the particularly pleasant property $H_{2n}(0) \neq 0$ has! As $H_{2n}(0)$ 0, $H_N(x) \not\equiv 0$, $N \geq 0$. One member of a family saves the entire family: The constant term of a Hermite polynomial of even degree is the saviour of the Hermite polynomials. Our analysis throws light on the usefulness of the Wronskian in obtaining the parity and recurrence relations satisfied by the Hermite polynomials. While it is good to know a better proof of a result, it is better to know good proofs of the same also. One may extend our approach to other real orthogonal polynomials associated with an even weight function and an interval of orthogonality symmetric with respect to the origin (e.g. Legendre polynomials and Chebyshev polynomials [1,2]).

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A NOTE ON THE MODULAR GROUP RING OF THE SYMMETRIC GROUP S_n

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Keyword : The mod p-envelp of a group.
Mathematics Subject Classification : 16A26 & 20E30.

In [2] the author has proved if G is a group of order two and K is a field with more than two elements then G^*, the mod p-envelope of G has a semi-group structure with a non trivial idempotent in it. It is interesting to note when G is the symmetric group S_n and K is a field having two elements 0 and 1 then G^* the mod p-envelope of G has a semi-group structure with non trivial idempotents in it and the number of elements in G^* is $2^{\lfloor n-1}$. For the sake of completeness we prove the following propositions. For definition's please refer [1] and [2].

Proposition 1. Let G be the symmetric group of permutations of (1,2,3) and K = (0,1) be the field of two elements. Then G^* is a semi-group with non trivial idempotents and the number of elements in G^* is $32 = 2^5 = 2^{\lfloor 3-1}$.

Proof. $G = \{ \begin{pmatrix}1&2&3\\1&2&3\end{pmatrix} = 1, \begin{pmatrix}1&2&3\\1&3&2\end{pmatrix} = p_1, \begin{pmatrix}1&2&3\\3&2&1\end{pmatrix} = p_2, \begin{pmatrix}1&2&3\\2&1&3\end{pmatrix} = p_3,$

$\begin{pmatrix}1&2&3\\2&3&1\end{pmatrix} = p_4, \begin{pmatrix}1&2&3\\3&1&2\end{pmatrix} = p_5 \}$.

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$$G^{*} = \{1+U\}$$

$$= \{1, p_1, p_2, p_3, p_4, p_5, 1+p_1+p_2, 1+p_2+p_3, 1+p_1+p_3,$$
$$1+p_1+p_4, 1+p_1+p_5, 1+p_2+p_4, 1+p_2+p_5, 1+p_3+p_4,$$
$$1+p_3+p_5, 1+p_4+p_5, p_1+p_2+p_3, p_1+p_2+p_4, p_1+p_2+p_5,$$
$$p_1+p_3+p_4, p_1+p_3+p_5, p_1+p_4+p_5, p_2+p_4+p_5, p_3+p_4+p_5$$
$$p_2+p_3+p_4, p_2+p_3+p_5, 1+p_1+p_2+p_3+p_4, 1+p_1+p_2+p_3+p_5,$$
$$1+p_1+p_2+p_4+p_5, 1+p_2+p_3+p_4+p_5, 1+p_1+p_3+p_4+p_5,$$
$$p_1+p_2+p_4+p_5+p_3\}.$$

Clearly G^{*} is a semi-group with identity

$$\{1+p_4+p_5, p_1+p_2+p_4, p_1+p_2+p_5, p_1+p_3+p_4, p_1+p_3+p_5,$$
$$p_2+p_3+p_3, p_2+p_3+p_5, 1+p_1+p_2+p_4+p_5, 1+p_1+p_3+p_4+p_5$$
$$\text{and } p_1+p_2+p_3+p_4+p_5\}$$

are nontrivial idempotents of G^{*}.

Proposition 2. Let G be the symmetric group of permutations of (1,2,3,4) and $K=(0,1)$ be the field of two elements. Then G^{*} is a semi-group with non trivial idempotents and the number elements in G^{*} is $2^{23} = 2^{\lfloor 4-1}$.

Proof. $G^{*} = \{1+U\}$

$$= \{\alpha = \Sigma\alpha_i g_i \;/\; \Sigma\alpha_i = 1$$
$$g_i \in G,\ \alpha_i \in (0,1)\}$$

$$|G^{*}| = 2[24C_1+24C_3+24C_5+24C_7+24C_9+24C_{11}]$$

$= 2^{23}$ (I have verified by calculations).

Consider

$$\alpha_1 = 1+\begin{pmatrix}1&2&3&4\\3&2&4&1\end{pmatrix}+\begin{pmatrix}1&2&3&4\\4&2&1&3\end{pmatrix}$$

$$\alpha_2 = 1+\begin{pmatrix}1&2&3&4\\2&3&1&4\end{pmatrix}+\begin{pmatrix}1&2&3&4\\3&1&2&4\end{pmatrix} \text{ and}$$

$$\alpha_3 = 1+\begin{pmatrix}1&2&3&4\\2&4&3&1\end{pmatrix}+\begin{pmatrix}1&2&3&4\\4&1&3&2\end{pmatrix}$$

are some of the non-trivial idempotents in G^*.

Promblem. Derive a method by which we can enumerate the number of non-trivial idempotents in G^*.

Theorem 3. Let G be the symmetric group of degree n. Let $K=(0,1)$ be the field of two elements, G^* the mod p envelope of G has $2^{\lfloor\underline{n-1}}$ elements and it has non-trivial idempotents and forms a semigroup.

Proof.

$$\begin{aligned} G^* &= \{1+U\} \\ &= \lfloor\underline{n}C_1 + \lfloor\underline{n}C_3 + \lfloor\underline{n}C_5 + \dots + \lfloor\underline{n}C_{\lfloor\underline{n-1}} \\ &= 2^{\lfloor\underline{n-1}}. \end{aligned}$$

Now every element in G^* is such that the sum of the coefficients is 1. This forces the number of terms of every element in G^* is odd. Clearly G^* is closed for the product of odd terms modulo two is also odd. Hence G^* is semi-group. $1 \in G^*$ so G^* is also a semi-group with identity. G^* has non-trivial idempotents for take.

$$\alpha = 1+\begin{pmatrix}1&2&3&4&\dots&n\\2&3&1&4&\dots&n\end{pmatrix}+\begin{pmatrix}1&2&3&4&\dots&n\\3&1&2&4&\dots&n\end{pmatrix}.$$

Clearly $\alpha^2=\alpha$ hence $\alpha \in G^*$ is a non trivial idempotent.

Example 1. Let $S_2 = \{\begin{pmatrix}1&2\\1&2\end{pmatrix}, \begin{pmatrix}1&2\\2&1\end{pmatrix}\}$, $K=(0,1,2)$ be the field of three elements. What is G^*?

Proof. $G^* = \{1 + U\}$

$= \{1, \begin{pmatrix}1&2\\2&1\end{pmatrix}, 2 + 2\begin{pmatrix}1&2\\2&1\end{pmatrix}\}.$

Order of $G^* = 3^{\lfloor 2 - 1} = 3$ and $2 + 2\begin{pmatrix}1&2\\2&1\end{pmatrix}$ is the non-trivial idempotent of G^*.

Example 1 can be extended to any field of p-elements

p a prime $G = \{ \begin{pmatrix}1&2\\1&2\end{pmatrix}, \begin{pmatrix}1&2\\2&1\end{pmatrix} \}$ and let

$$K = (0, 1, 2, \ldots, p-1)$$

$$G^* = \{1, (p-1)\begin{pmatrix}1&2\\2&1\end{pmatrix} + 2, 2\begin{pmatrix}1&2\\2&1\end{pmatrix} + p - 1, (p-2)\begin{pmatrix}1&2\\2&1\end{pmatrix} + 3, 3\begin{pmatrix}1&2\\2&1\end{pmatrix} + p - 2, \ldots, \frac{p+1}{2} + \frac{p+1}{2}\begin{pmatrix}1&2\\2&1\end{pmatrix}\}.$$

G^* is a semi-group with p-elements, and $\frac{p+1}{2} + \frac{p+1}{2}\begin{pmatrix}1&2\\2&1\end{pmatrix}$ is an idempotent of G^*. Then order of

$$G^* = p^{\lfloor 2 - 1} = p.$$

Hence we generalize this as a problem.

Problem. If G is symmetric group of order n, $K = (0, 1, 2, \ldots, p-1)$ be a semi-group of order $p^{\lfloor n - 1}$ and has non-trivial idempotents.

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AN ALTERNATIVE APPROACH TO TOLLMIEN'S ENERGY SPECTRA DERIVED FROM HEISENBERG'S EQUATION OF TURBULENT FLOW

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ABSTRACT

Tollmien (1952-53) had to work with experimenental data of windtunnel measurements due to Stewart and Townsend (1951) for the evaluation of turbulent energy spectra. In this paper similar categories of spectra have been shown to be alternatively derivable from Sen's (1951) self preserving solution of Heisenberg's (1948) equation and by extending numerical tables of Ghosh (1955) the author has ultimately obtained self-similar spectrum curves somewhat analogous to the normalized apectra calculated by Tollmien at different times of turbuleNt fluid flow.

1. INTRODUCTION

It is well known that experimental studies of turbulence are based on relevant data obtained experimentally at least at a distance of 20 mesh length behind the grids of wind tunnel. Tollmien's numerical evaluation of turbulent energy spectra (1952-53)at times t=0 t=0.2, t=0.5,t=1 has a footing on experimental results at 30mesh length. If F(K,t) be the energy spectrum function of homogeneous- isotropic turbulence for wave number K at time t,the decay of turbulence occurs according to Heisenberg's (1948) equation

$$-\frac{\partial}{\partial t}\int_{\infty}^{k} F(k,t)dk=2(\nu+n_k)\int_{\infty}^{k} k^2F(k,t)dk \qquad \dots (1)$$

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where ν is kinematic viscosity and n_k is the turbulent viscosity given by

$$n_k = \nu_H \int_k^{\infty} \frac{d\ F(K,t)}{k^3}\, dk$$

ν_H being a constant.

Since the integro-differential equation (1) contains only the first order time derivative, is follows that its solution is uniquely determined by imposing initial condition of the form

$F(K,t)|_{t\to 0} = F(K,O) = F_o(K)$, (say).

Therefore, if the form of the spectrum at initial time is known from experiment, one can determine the spectrum for all subsequent times by numerical integration Thist type of calculation has been incidentally carried out by Tollmien for Heisenberg's equation (1) and the data of Stewart and Townsend(1951). Tollmien took $F_o(K)$ in the form of data of Fig.(a) and $Y_H=0.45$ according to Proudman's estimate (1951). Furthermore, Tollmien attempted to normalize his solution in accordance with the data of Fig.(a) so that the values of F(K,t) for large K were as near time-independent as possible. The curves obtained in this way vide Fig.(b) were found to be qualitattively similar to empirical curves of Stewart and Townsend shown in Fig. (a). Although it has been stated that the resulting function F(K,t) did not display self preserving behaviour in time, yet the current evaluation will lead to inference otherwise if we start with Ghosh's (1955) numerical tables.

Nevertheless, turbulent energy calculated by Tollmien was found to be in very good agreement with the empirical "minus one law" right upto t=0.5 and this law was usually regarded as an evidence for self-preservation.

In tins context it may be mentioned that Karman-Howarth(1938) defined self preserving solutions as to be those whose curves do not undergo a change of shape with time, the only change being the change of scale. The motion in a sense remains similar to itself.

Heisenberg has shown that the equation (1) admits of a self similar solution of the form for

$$F(k,t) = \frac{1}{\sqrt{t}}\, f(K\sqrt{t}) \qquad \ldots (2)$$

for which f(x) with x=kt obeys an ordinary differential equation derivable from (1). This set of self-similar solutions was numerically calculated by Chandrashekhar (1949). Such numerical solutions so obtained on the basis of equation (2) have obvious break of similarity at the left end of self similar curves [cf. Karman & Lin (1951)].

In order to overcome this drawbhack Sen(1951) suggested a bunch of solutions of Heisenberg's equation (1) in the form

$$F(K,t) = \frac{S^3}{\nu^2 k_o^3 t_o^3 \tau^2}\, f\left(S\,\frac{K}{K_o}\right) \qquad \ldots (3)$$

where $s=s(\tau)$, $T=t/t_o$ and K_o, t_o are constant. Using Reynolds number R as

$$\frac{1}{R} = \nu k_o^2 t_o \text{ and } SK/K_o = x,$$

the equation of f(x) is obtined by substitution of (3) in (1) as

$$2(1-c)\int_0^x f(x)dx - cxf(x) = 2\int_x^\infty \sqrt{\frac{f(x)}{x^3}}\, dx \int_0^x x^2 f(x)dx + \frac{2\tau}{S^2R} \qquad \ldots (4)$$

where c is parameter introduced by Sen as $c = \frac{1}{s}\frac{ds}{dt}$ i.e. $s = \alpha\tau^c$, α being a constant.

Neglecting the last term of equation (4) for early period of decay process where Reynolds number of fluid flow is infinitely large, Ghosh has obtained numerical tables by means of Sen's self similar solutions.

$$F(k,t) = \left[\frac{1}{\nu_H^2 k_o^3 t_o^2} \left(\frac{t}{t_o}\right)^{3c-2} f\left(\frac{k}{k_o} \left\{\frac{t}{t_o}\right\}\right)\right]_{c=\frac{2}{7}}$$

with asymtotic behaviours

$$f(x) \approx x^4, (x \to 0) \text{ and } f(x\uparrow x^{-5/3}, (x \to \alpha).$$

In doing so Ghosh also followed Chandrasekhar for introduction of two auxiliary functions as

$$g = x^3 f(x) \text{ and } y(x) = \int_o^x x^2 f(x) dx$$

to recast the equation (4) for $R \to \alpha$ into the form

$$cg^{3/2} g'' + (4+g')y + 2g^{1/2}(2-cg') - cg) - 4g = 0 \qquad \ldots (5)$$

where $g' = \frac{dg}{dy}$.

If $c=\frac{1}{2}$, the relation (5) gives Chandrasekhar's equation for the preparation of numerical table of $(y, g(y))$ and thereafter $(x, f(x))$, whereas $c = \frac{2}{7}$ leads to Ghosh's tables for $(y, g(y)]$, and consequently $(x, f(x))$. It is observed that desipite $(x, f(x))$ curves of Chandrasekhar's tabular values and those due to Ghosh are by and large similar in pattern, yet it is obvious that there is a break of similarity at the left end portion of each individual $(x, f(x))$ curves due to Chandrasekhar's tables. This is because of the fact that the asymptotic behaviours of Chandrasekhar's tables involve $f(x) \approx x$, $(x \to o)$ and $f(x) \approx x^{-5/3}$, $(x \to \alpha)$ of which, the first one i.e. $f(x) \approx x, (x \to 0)$ leads to break of similarity while $F(K,t)|_{t \to o}$ is to be accounted for through a behaviour of $f(x)$ at $x \to 0$. Hence to proceed with this form for $f(x) = f\left\{\frac{k}{k_o} \sqrt{\frac{t}{t_o}}\right\}$ as $t \to 0$ is anomalous.

As such in the present paper it has been attempted to generate numerical values of F(K,t) when $t \rightarrow 0$ as obtainable from Sen's self-similar solution of equation (1) for parametric value $c=\frac{2}{7}$ by making an extension of Ghosh's numerical results transformed into new tables that eventually yield identical curves resembling Tollmien's spectra and by dint of this approach it became possible to maintain the self preserving behaviour throughout the entire specturm as well.

2. APPROACH TO THE POBLEM

In the present problem the equation (1) has, in fine, the more general solution of the from

$$F(k,t) \approx \text{const.}\, t^{3c-2} f(kt^{c})$$

where c is an arbitarary constant.

For $c=\frac{2}{7}$, $(0<c<\frac{2}{3})$ we have in particular

$$F(k,t) \simeq \frac{1}{\nu_H^2 K_o^3 t_o^2} \left(\frac{t}{t_o}\right)^{-8/7} f\left(\frac{k}{k_o}\left\{\frac{t}{t_o}\right\}^{2/7}\right) \qquad \ldots (6)$$

Evidently asymptotic behaviours of F(k,t) are here found to be

$$F(k,t) \approx k^{4}, (k \rightarrow 0) \quad F(k,t)=k^{-5/3}, (k \rightarrow \infty)$$

The first one of the asymptotes was supported in the works of C.C. Lim (1947) and G.K. Batchelor (1949), while the other asymptotic behaviour leads to Kolmogorov (1941) spectrum. In addition to theoretical background, in fact the spectrum K^4, $(k \rightarrow 0)$ has got also strong experimental confirmation according to Debnath (1978) and the evaluation of F(k,t) for $c = \frac{2}{7}$ concerns in this case the derivation of numerical solutions of (x,f(x)) without any break of self-similarity at $x \to o$ i.e. $t \to o$ or $k \to o$ as well as $t \to o$. According to our requirement we are to set forth the intial conditions as to be

$$F(k,t)\Big|_{t \to o} = \frac{1}{\nu_H^2 K_o^3 t_o^2} \left(\frac{t}{t_o}\right)^{-8/7} [f(x)]_{x \to o} \text{ for } x = \frac{k}{k_o}\left(\frac{t}{t_o}\right)^{2/7} \quad \ldots (7)$$

where $(x, f(x))$ data are to be collected from numerical tables of Ghosh and K_o, t_o being unity for the present instance. The reliability of new set of results may be verified by the condition $F(k,t) = \Lambda k^4$ as $k \to o$, where Λ is Loitsiansky's (1939) invariant in respect of the tables concerned

3. NUMERICAL TABLES AND GRAPHS

In the light of aforesaid facts we have developed the following numerical tables I & II where first, second columns represent data taken from Ghosh's tables and for $x = \frac{k}{k_o}\left(\frac{t}{t_o}\right)^{2/7}$ the numerical values of third, fourth, fifth columns of each table have been calculated at t=0.5, 0.2, 0.1 respectively. On the basis of such extended tables we have generated turbulent energy spectrum converted finally in terms of curves (Fig.1 and Fig.2) which are found to be akin to the Tollmien's spectra and thereby leading to remarkable modification in the self preserving nature of spectra as pictured by the current approach.

CONCLUDING REMARKS

(i) Thus to start with fourth power law $f(x) \simeq x^4$, $(x \to 0)$ meaning $x = \frac{k}{k_o} \left(\frac{t}{t_o}\right)^{2/7} \to 0$ includes the behaviour at $t \to 0$ and hence having regard to Sen's (1951) self similar solution of Heisenberg's (1948) energy decay equation one can safely work with the tables of Ghosh (1955) in lieu of wind-tunnel data of Stewart and Townsend (1951) for the evaluation of turbulent energy spectrum at different times where the resulting function $F(k,t)$ exhibits self preserving behaviour with time as evident from the graph of the present note.

(ii) One may extrapolate freely at the end of the tables extended in the current work as there is no break of self similarity anywhere in the whole range of the spectra and such extrapolation at the extreme upper end of the tables concerns the evaluation of Loitsianky's constant Λ that remains independent of time.

(iii) At the initial stage of decay phenomean when low frequency part of the spectrum plays significant role or in otherwords when the low frequency eddies are predominant, then variety of spectra satisfying similarity law may be possible depending on the method of production of turbulence as the parameter c being proposed to be associated with the mode of excitation of turbulence and this approach is valid at a stage of turbulence of negligible viscous dissipation by the wave numbers concerned so that the inertia terms of Navier-stokes equation are mainly responsible for the entire transfer of energy down the spectrum of such self preserving isotropic turbulence.

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TABLE : 1

x	f(x)	k	k	k
0.3737	0.1314	0.4555	0.5919	0.7215
0.4166	0.1660	0.5078	0.6598	0.8043
0.4708	0.2067	0.5739	0.7456	0.9090
0.4910	0.2203	0.5985	0.7777	0.9480
0.5320	0.2442	06485	0.8426	1.0271
0.5656	0.2592	0.6895	0.8958	1.0920
0.6623	0.2792	0.8074	1.0490	1.2787
0.7360	0.2720	0.8972	1.1657	1.4210
0.8006	0.2525	0.9759	1.2680	1.5457
0.8607	0.2279	1.0492	1.3632	1.6618
0.9200	0.1979	1.1215	1.4571	1.7762
0.9810	0.1664	1.1959	1.5537	1.8940
1.0466	0.1355	1.2758	1.6576	2.0207
1.1206	0.1003	1.3660	1.7748	2.1635
1.2111	0.0668	1.4763	1.9182	2.3383
1.8446	0.0037	2.2486	2.9215	3.5614

TABLE : II

x	f(x)	k	k	k
0.3736	0.1320	0.4554	0.5917	0.7213
0.4160	0.1695	0.5071	0.6589	0.8032
0.4687	0.2158	0.5714	0.7423	0.9049
0.4882	0.2323	0.5951	0.7732	0.9426
0.5273	0.2628	0.6428	0.8351	1.0181
0.5582	0.2845	0.6805	0.8841	1.0777
0.6469	0.3295	0.7886	1.0246	1.2490
0.7109	0.3452	0.8666	1.1259	1.3725
0.7638	0.3481	0.9311	1.2097	1.4747
0.8104	0.3436	0.9879	1.2835	1.5646
0.8530	0.3346	1.0398	1.3510	1.6469
0.9310	0.3083	1.1349	1.4745	1.7975
1.0042	0.2757	1.2241	1.5905	1.9388
1.0759	0.2397	1.3115	1.7040	2.0772
1.1492	0.2019	1.4009	1.8201	2.2186
1.2270	0.1633	1.4957	1.9433	2.3690
1.3137	0.1245	1.6014	2.0807	2.5364
1.4168	0.0861	1.7271	2.2440	2.7354
1.5547	0.0490	1.8952	2.4624	3.0016
1.8078	0.0147	2.2037	2.8632	3.4903
2.3858	0.0009	2.9083	3.7787	4.6063

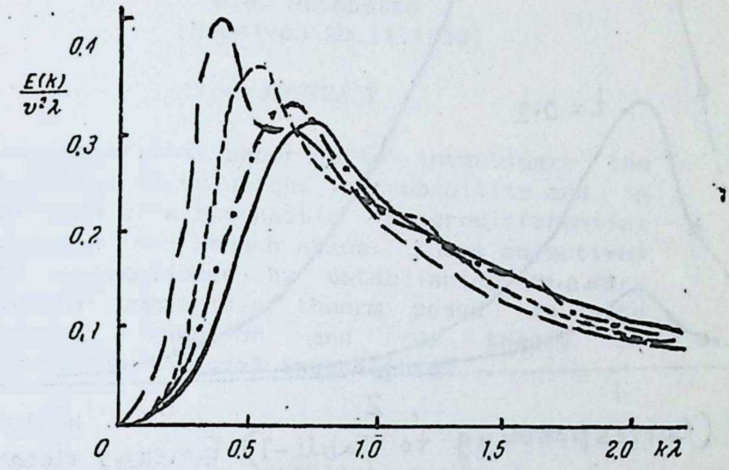

FIG. (a) Normalized spectral density functions at different distances from the grid [Stewart and Townsend (1951)].

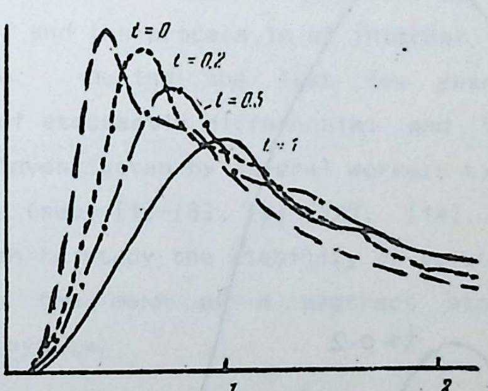

FIG. (b) Normalized spectra $F(k,t)$ for different t, calculated by Tollmien (1952 - 1953).

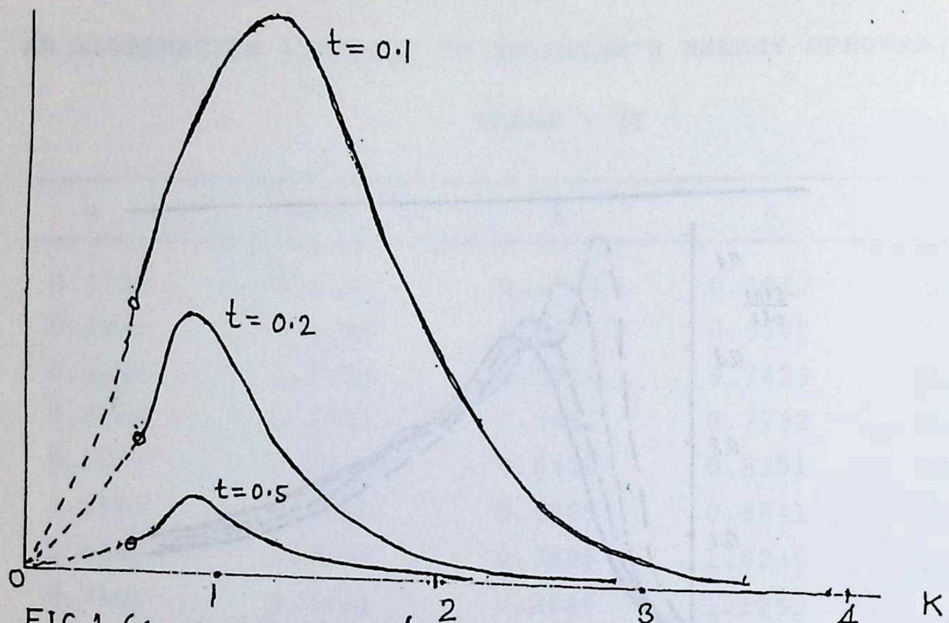

FIG.1 (Corresponding to Table-I) Energy decay spectra F(k,t).

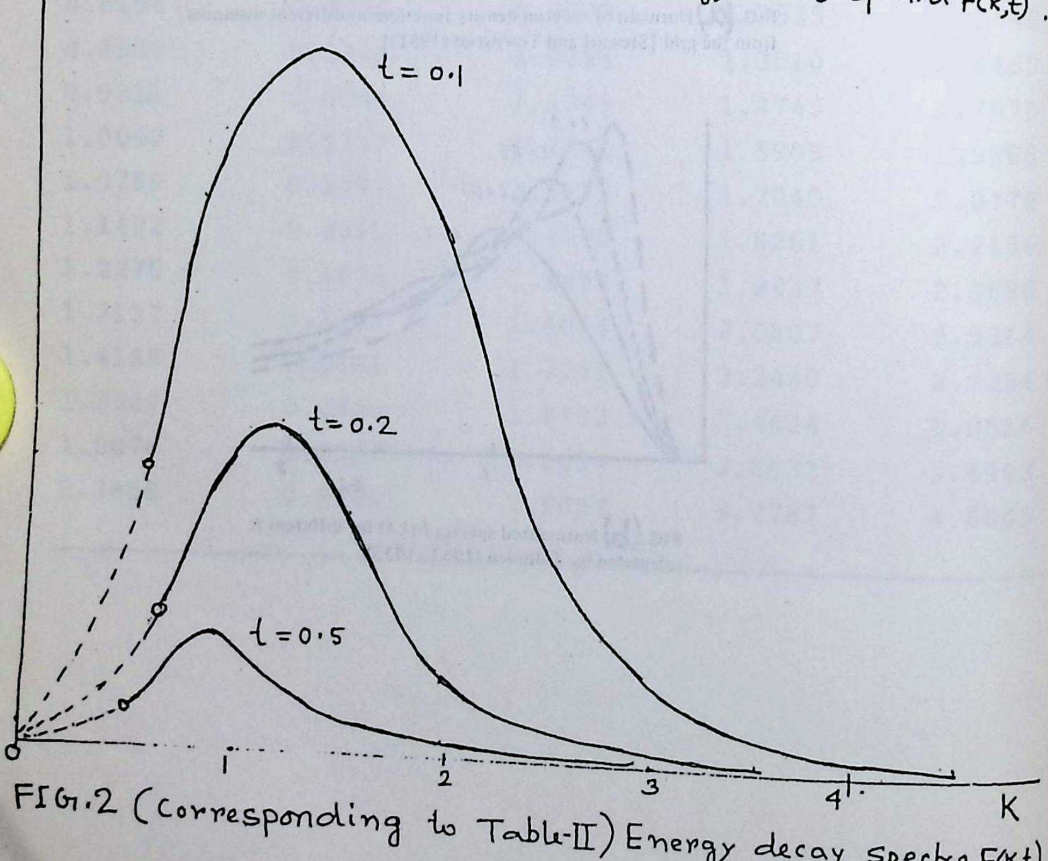

FIG.2 (Corresponding to Table-II) Energy decay spectra F(K,t).

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ON THE STABILITY OF A STOCHASTIC INTEGRODIFFERENTIAL EQUATIONS IN A BANACH SPACE

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ABSTRACT

The aim of this paper is to investigate the stability of solutions in probability and in the mean of a stochastic integrodifferential equation in a Banach space. These objectives are accomplished by establishing a very general comparison theorm based on the Lyapunov function and the theory of stochastic integral inequalities.

1. INTRODUCTION

Stochastic stability problems occur in almost all phases of physics, control theory, numerical analysis and economics where dynamical models subject to random disturbances appear, and the process is of interest over a long period of time. During the last few years, the stability analysis of stochastic differential and integral equations has been investigated by several workers by using different techniques (see, [1]-[5], [8]-[12], [14], [15]). In this paper we wish to study the stability of solutions in probability and in the mean of a abstract stochastic integrodifferential system.

$$(1)\quad x'(t,\omega) = A(t,\omega)x(t,\omega)+f(t,x(t,\omega),\int_{t_0}^{t}(K)[t,s,x(s,\omega),\omega]ds,\omega)$$

$$+g(t,x(t,\omega),\omega),\quad x(t_0,\omega) = x_0(\omega),$$

given certain connective information concerning the solutions of the corresponding abstract stochastic integrodifferential system

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$$(2)\quad y'(t,\omega) = A(t,\omega)y(t,\omega)+f[t,y(t,\omega), \int_{t_0}^{t} K[t,s,y(s,\omega),\omega]ds,\omega],$$

$$y(t_0,\omega) = x_0\ (\omega),$$

and the auxiliary stochastic integrodifferential equation

$$(3)\quad r'(t,\omega)= H_1(t,r(t,\omega),\int_{t_0}^{t} K_0[t,s,r(s,\omega),\omega]ds,\omega)+H_2(t,r(t,\omega),\omega),$$

$$r(t_0\omega) = r_0(\omega),$$

where

(i) $\omega \in \Omega$ the supporting set of a complete probability measure space (Ω,F,P);

(ii) $x(t,\omega)$, $y(t,\omega)$ are the unknown random processes which map $R^+ \times \Omega$ into X, where X is a seperable Banach space with norm $\| \,.\, \|$;

(iii)for each $t \in R^+$, the set $A(t,\omega)$ is a family of random operators [1] in X, with domain $D[A(t,\omega)]$;

(iv) $g(t,x,\omega)$, $K[t,s,x,\omega]$ and $f(t,x,\bar{x},\omega)$ are Borel measurable maps from $R^+ \times X \times \Omega$, $R^+ \times R^+ \times X \times \Omega$ and $R^+ \times X \times X \times \Omega$ into X respectively such that $g(t,0,\omega) = 0$, $f(t,0, \int_{t_0}^{t}(K)$ $[t,s,0,\omega]ds,\omega) = 0$;

(v) $H_2(t,r,\omega)$, $K_0[t,s,r,\omega]$ and $H_1(t,r,\bar{r},\omega)$ are Borel measurable maps from $R^+ \times R \times \Omega$, $R^+ \times R^+ R \times \Omega$ and $R^+ \times R \times R \times \Omega$ into R respectively such that $H_2(t,0,\omega) = 0$, $H_1(t,0,\int K_0[t,s,0,\omega], \omega) = 0$.

Recently, by assuming the existence of solutions of stochastic differential and integral equations Kats and Krasovskii [2], Martinjuk [8], Ladde [4], Ladde, Lakshmikantham and Liu [5], Rao and Manougian [14], Pachpatte [10]-[12] and others have studied the stability and boundedness of solutions by employing the second method of Lyapunov and the theory of differential and integral inequalities. Here, by assuming the existence of global mean square continuous solutions of (1), (2) and (3) for all $t \geq t_0$ and employing Lyapunov functions and the theory of integral inequalities we give sufficient conditions for the stability in the probability and in the mean of the solutions of (1) by means of comparison with the connective stablility in the probability and in the mean of (2) and (3).

2. PRELIMINARIES

In our subsequent discussion, we are interested in random linear operators difined on subsets of $X \times \Omega$. Suppose that for each $t \in R^+$, $A(t,\omega)$ is a closed random linear operator such that $[I-\alpha A(t,\omega)]^{-1}$ exists for a real α, where I is the identity operator. Suppose further that $D[I-\alpha A(t,\omega)]^{-1}$ is dense in X. Denote the resolvant of $A(t,\omega)$ by $R(\alpha, A(t,\omega)) \equiv [I\alpha A(t,\omega)]^{-1}$. Then the following results are known [1]:

(i) $R(\alpha,A(t,\omega))$ is a random linear operator for each $t \in R^+$;

(ii) $D[R(\alpha,A(t,\omega))] = X$;

(iii) $[I-\alpha A(t,\omega)]\ R(\alpha,A(t,\omega))x = x$ for $x \in X$;

(iv) $R(\alpha,A(t,\omega))\ [I-\alpha A(t,\omega)]x = x$ for $x \in D\ [A(t,\omega)]$.

We shall use the following definitions in our subsequent discussion, (see,[4, 5, 9, 12]).

Definition 1. Let $x(t,\omega) \in D[A(t,\omega)]$ be a random variable with values in X for each $t \in R^+$. Then $x(t,\omega)$ is called a solution of (1) if $x(t,\omega)$ has a strong derivative in X and satisfies (1).

Definition 2. The trivial solution of (1) is said to be (SP_1) *Stable in probability*, if for each $\varepsilon > 0$, $\eta > 0$, $t_0 \in R^+$, there exists $\delta = \delta(t_0,\varepsilon,\eta) > 0$ such that

$$P\{\omega : \|x_0(\omega)\| > \delta\} < \eta$$

implies

$$P\{\omega: \|x(t,\omega)\| \geq \varepsilon\} < \eta,\ t \geq t_0;$$

(SP_2) *asymptotically stable in probability*, if it is stable in probability and, if for any $\varepsilon > 0$, $\eta > 0$, $t_0 \in R^+$, there exist positive number $\delta_0 = \delta_0(t_0)$ and $T = T(t_0,\varepsilon\eta)$ such that

$$P\{\omega : \|x_0(\omega)\| > \delta_0\} < \eta$$

implies

$$P\{\omega : \|x(t,\omega)\| \geq \varepsilon\} < \eta,\ t \geq t_0 + T;$$

(SM_1) *Stable in the mean*, if for each $\varepsilon > 0$, $t_0 \in R^+$, there exists a $\delta = \delta(t_0,\in) > 0$ such that the inequality

$$E[\|x_0(\omega)\|] \leq \delta$$

implies

$$E[\|x(t,\omega)\|] < \varepsilon,\ t \geq t_0;$$

(SM_2) *asymptotically stable in probability*, if it is stable in the mean and, if for any $\varepsilon > 0$, $t_0 \in R^+$, there exist positive numbers $\delta_0 = \delta_0(t_0)$ and $T=T(t_0,\varepsilon)$ such that the inequality

$$E[\|x_0(\omega)\|] \leq \delta_0$$

implies

$$E[||x(t,\omega)||] < \varepsilon, \ t \geq t_0+T.$$

Based on Definition 2, one can formulate other difinitions of stability and boundedness (see, [4,5,14]) analogously.

Defination 3. A function b (r) is said to belong to the class K if $b \in C[R^+ + R^+]$, $b(0) = 0$, $b(r)$ is strictly increasing in r.

Definition 4. A function $b(r)$ is said to belong to the class VK, if $b \in C\ [R^+, R^+]$, $b(0) = 0$, $b(r)$ is a convex and strictly increasing in r.

Definition 5. A function $a(t,r)$ is said to belong to the class CK if $a \in C(R^+ X R^+, R^+]$, $a\ (t,0) = 0$, and $a(t,r)$ is concave and increasing in r for each fixed $t \in R^+$.

In order to avoid monotonicity, hereafter, it will be understood, unless otherwise specified, that the functions involved in (1)-(3) satisfy the conditions (iv) and (v) given in Section 1, and furthermore all equalities, inequalities, and relations that involve the random processes will hold with probability 1.

3. COMPARISON THEOREM

In this section, we develop a very general comparison theorem for stochastic integrodifferential system (1) which plays an important role in our subsequent discussion. In order to establish our main result in this section, we require the following hypothesis:

(H) the solution $y(t,t_0,x_0,\omega)$ of (2) exist for all $t \geq t_0$, unique, continuous with respect to the initial data and $||y(t,t_0,X_0,\omega)||$ is locally Lipschitzian in x_0.

THEOREM 1. *Assume that*

(i) *for each* $t \in R^+$, $x \in X$,

$$(4) \quad \lim_{h\to 0^+} R(h,A(t,\omega))x = x,$$

(ii) *the hypothesis (H) holds,*

(iii) there exists a measurable map $V(t,x,\omega)$ defined on $R^+ \times X \times \Omega$ into R^+ such that $V(t,x,\omega)$ is locally Lipschitzian in $(t,x) \in R^+ \times X$ ω.p.1,

(iv) *for* $t \in R^+$, $x \in X$ and $\omega \in \Omega$,

$$(5) \quad D^+V(s,y(t,s,x(s,\omega),\omega),\omega)$$

$$= \lim_{h\to 0^+}\sup \frac{1}{h}\,[V(s+h,y(t,s+h,R(h,A(s,\omega))x(s,\omega)$$

$$+h\{f(s,x(s,\omega),\int_{t_0}^{s}K[s,\tau,x(\tau,\omega),\omega]d\tau,w)$$

$$+ g(s,x(s,\omega),\omega)\},\omega),\omega)$$

$$- V(s,y(t,s,x(s,\omega),\omega),\omega)]$$

$$\leq H_1(s,V(s,y(t,s,x(s,\omega),\omega),\omega),\int_{t_0}^{s}K_0[s,\tau,V(\tau,Y(t,\tau,x(\tau,\omega),\omega)\omega]$$

$$d\tau,\omega) + H_2(s,V(s,y)(t,s,x(s,\omega),\omega),\omega),\omega),$$

where $H_2(t,r,\omega)$, $K_0[t,s,r,\omega]$ *and* $H_1(t,r,\bar{r},\omega)$ are as defined in section 1,

(v) the maximal solution $r(t,t_0,r_0\omega)$ of (3) exists for $t \geq t_0$. Then $x(t,\omega) = x(t,t_0,x_0,\omega)$ is any solution of (1) we have

$$(6) \quad V(t,x(t,\omega),\omega) \leq r(t,t_0,r_0,\omega), \; t \geq t_0,$$

provided

$$(7) \qquad V(t_0, y(t,t_0,x_0,\omega),\omega) \le r_0(\omega).$$

Proof. Define

$$m(s,\omega) = V(s,y(t,s,x(s,\omega),\omega),\omega) \text{ for } t_0 \le s \le t,$$

so that

$$m(t_0,\omega) = V(t_0,y(t,t_0,x_{0},\omega),\omega).$$

Since $x(t,\omega)$ is a sample solution of (1) and $V(t,x,\omega)$ satisfies hypothesis (iii), we conclude that $m(s,\omega)$ is sample absolutely continuous W.p.1 for $t \ge t_0$ (see [4] and a particular reference given therein). For small $h > 0$, we have

$$
\begin{aligned}
(8)\quad & m(s+h,\omega)-m(s,\omega) \\
&= V(s+h,y(t,s+h,x(s+h,\omega),\omega),\omega) - V(s,y(t,s,x(s,\omega),\omega),\omega) \\
&= V(s+h,y(t,s+h,x(s+h,\omega),\omega),\omega - V(s+h,y(t,s+h,R(h,A(s,\omega))x(s,\omega) \\
&\quad +h\{f(s,x(s,\omega),\int_{t_0}^{s} K[s,\tau,x(\tau,\omega),\omega]d\tau,\omega) + g(s,x(s,\omega),\omega)\},\omega),\omega) \\
&\quad + V(s+h,y(t,s+h,R(h,A(s,\omega))x(s,\omega) \\
&\quad +h\{f(s,x(s,\omega),\int_{t_0}^{s} K[s,\tau,x(\tau,\omega),\omega]d\tau,\omega) + g(s,x(s,\omega),\omega)\},\omega),\omega) \\
&\qquad -V(s,y(t,s,x(s,\omega),\omega),\omega) \\
&\le L||y(t,s+h,x(s+h,\omega),\omega)-y(t,s+h,R(h,A(s,\omega))x(s,\omega) \\
&\quad +h\{f(s,x(s,\omega),\int_{t_0}^{s} K[s,\tau,x(\tau,\omega),\omega)]d\tau,\omega)+g(s,x(s,\omega),\omega)\},\omega)|| \\
&\quad + V(s+h,y(t,s+h,R(h,A(s,\omega))x(s,\omega)+h\{f(s,x(s,\omega), \\
&\quad \int_{t_0}^{s} K[s,\tau,x(\tau,\omega),\omega]d\tau,\omega) + g(s,x(s,\omega),\omega)\},\omega),\omega)
\end{aligned}
$$

$$-V(s,y(t,s,x(s,\omega),\omega),\omega)$$

$$\le LM||x(s+h,\omega) - R(h,A(s,\omega))x(s,\omega) -h\{f(s,x(s,\omega),$$

$$\int_{t_0}^{s} K[s,\tau,x(\tau,\omega),\omega]d\tau,\omega) + g(s,x(s,\omega),\omega)\}||$$

$$+V(s+h,y(t,s+h,R(h,A(s,\omega))x(s,\omega) + h\{f(s,x(s,\omega),$$

$$\int_{t_0}^{s} K[s,\tau,x(\tau,\omega),\omega]d\tau,\omega) + g(s,x(s,\omega),\omega)\},\omega),\omega)$$

$$-V(s,y(t,s,x(s,\omega),\omega),\omega),$$

where L and M are positive constants. Since $x(t,\omega) \in D[A(t,\omega)]$, we have

$$R(h,A(t,\omega))\,[I-hA(t,\omega)]x(t,\omega) = x(t,\omega).$$

Hence

$$(9)\quad R(h,A(t,\omega))x(t,\omega) = x(t,\omega) + hA(t,\omega)x(t,\omega)$$
$$+h[R(h,A(t,\omega))A(t,\omega)x(t,\omega)-A(t,\omega)x(t,\omega)].$$

Using (9) in (8) we have

$$(10)\quad m(s+h),\omega\ )-m(s,\omega\ ) \le LM||\ x(s+h,\omega)-x(s,\omega)-hA(s,\omega)x(s,\omega)$$
$$-h\ [R(h,A(s,\omega))A(s,\omega)x(s,\omega)-A(s,\omega)x(s,\omega)]\ \ -h\ \{f(s,x(s,\omega),$$
$$\int_{t_0}^{s} K\ [\ s,\tau,x\ (\tau,\omega),\omega,\ \omega)\ +g(s,x(s,\omega),\omega)\}||+V(s+h,y(t,s+h,$$
$$R(h,A(s,\omega))\ x(s,\omega)+h\ \{f(s,x(s,\omega),\ \int_{t_0}^{s} K\ [s,\tau,x(\tau,\ \omega),\omega)]d\tau,\omega)$$
$$+g(\ s,x(s,\omega)\omega\},\ \omega),\ \omega),-\ V(s,y(t,s,x(s,\omega),\omega),\omega).$$

Now, using (10),(1) and the assumption given in the statement of the theorem, and sample absolute continuity of $m(s,\omega)$ yeilds the inequality

$$(11)\ m'(s,\omega) \leq H_1(s, m(s,\omega), \int_{t_0}^{s} K_0[s,\tau,m(\tau,\omega)\ \omega]d\tau, \omega\ )$$

$$+ H_2(s, m(s,\omega), \omega)$$

almost everywhere on $(s,\omega) \in R^+ \times \Omega$. From (7) and the definition of $m(s,\omega)$ we have $m(t_0,\omega) \leq r_0\ (\omega)$. Now by following the standard argument analogous to deterministic Theorem 1 in Ref [13, p.192] with suitable modification yields

$$(12)\ m(s,\omega) \leq r(s,t_0,r_0,\omega),\ t_0 \leq s \leq t$$

Provided $m(t_0,\omega) \leq r_0(\omega)$. Since

$$m(t,\omega) = V(t,y(t,t,x(t,\omega),\omega),\omega) = V(t,x(t,\omega),\omega),$$

the desired result (6) follows from (12) by setting $s = t$. The proof is complete.

It is interesting to make the following observations ·

(a) Taking $r_0(\omega) = V(t,t_0,x_0,\omega),\omega)$, the inequality (6) becames

$$(13)\ V(t,x(t,\omega),\omega) \geq r(t,t_0,V(t_0,y(t,t_0,x_0,\omega),\omega),\omega), \quad t \geq t_0,$$

which shows the connection between the solutions of systems (1) and (2) in terms of the maxial solutions of (3).

(b) The trivial function

$$A(t,\omega)y(t,\omega) + f(t,y(t,\omega), \int_{t_0}^{t} K\ [t,s;y(s,\omega),\omega\ ]ds,\ \omega) = 0$$

is admissible in Theorem 1 to yield the estimate (6) provded $V(t_0,x_0,\omega) \leq \gamma_0(\omega)$. In this case $y(t,t_0,x_0,\ \omega) = x_0$ and thus the hypothesis (H) is trivially verified. Since $y(t, t, x, \omega) = x$, the difinition (5) of $D^+V(s,y(t,s,x(s,\omega),\omega),\omega)$ reduces to

$$(14)\quad D^+V(t,x(t,\omega),\omega),\omega) = \lim_{h\to 0^+} \sup \frac{1}{h}[V(t+h,R(h,A(t,\omega)x(t,\omega) + h\{f(t,x(t,\omega), \int_{t_0}^{t} K[t,s,x(x,\omega),\omega]ds, \omega) + g(t,x(t,\omega),\omega)\},\omega)-V(t,x(t,\omega),\omega)],$$

which is the usual definition of the generalized derivative of the Lyapunov function relative to the system (1).

We note that Theorem 1 is a direct eextension to stochastic integrodiffeerential systems of the corresponding comparision theorem for deterministic equations recently established by the present author in [13, Theorem 3] which in turn is a further extension of the comparison theorem established by Ladde, Lakshmikantham and Leela in [6]

4.STABILITY RESULTS

In this section employing the comparison theorem developed in the preceding section, we shall present various results giving sufficient conditions for stability in probability and in the mean of the trivial solution of the stockastic integro-differential system (1). Before giving the main results in this section, relative to the stochastic integrodifferential system (2) and the auxiliary stochastic integrodifferential equation (3) we need the corresponding definitions (CSP_1^*) (CSP_2^*),$(CSM_1^*)$and (CSM_2^*) in our discussion.

For example the connective stability in probability (CSP^*) runs as follows (see, [6], for a similar definition in deterministic case).

Definition 6: The trivial solutions of (2) and (3) are said to be (CSP_1^*) connectively stable in probability, if for each $\varepsilon > 0$, $\eta >. 0$, $t_0 \in R^+$, there exists a $\delta = \delta(t_0;\varepsilon\ \eta) > 0$ such that

$$P\ \{\omega : ||x_0\ (\omega)\ || > \delta\} < \eta$$

implies

$$P\ \{\omega : r(t,\ t_0,\ v(t_0, y(t, t_0, x_0, \omega), \omega), \omega \geq \varepsilon\} < \eta, t \geq t_0.$$

Other definitions may similarly be formulated.

THEOREM 2. *Assume that the hypotehses of Theoreem 1 hold and*

(15) $b(||x||) \leq V(t, x(t,\omega), \omega) \leq a(t,\ ||x||)$,

where $b,\ a(t,.) \in \varkappa$. *Then*

(i) (CSP_1^*) *implies* (SP_2),

(ii) (CSP_2^*) *implies* (SP_1)

Proof. First we prove the statement (i). Let $\eta > 0$, $\varepsilon > 0$, and $t_0\ \varepsilon\ R^+$ be given. Assume that (CSP_1^*) holds. Then, given $b(\varepsilon) > 0$, $\eta > 0$, and $t_0 \in R^+$, there exists a positive function $\delta_1 = \delta_1 (t_0, \varepsilon, \eta)$ such that

(16) $P\ \{\omega : r(t, t_0, V(t_0, y(t, t_0, x_0, \omega), \omega), \omega) \geq b(\varepsilon)\} < \eta, t \geq t_0$

provided

(17) $P\{\omega : ||x_0(\omega)|| > \delta_1\} < \eta$.

Let us choose $r_0(\omega)$ so that $V(t_0, y(t, t_0, x_0, ., \omega), \omega) = r_0(\omega)$ and

$$r_0(\omega) = a(t_0, ||\ x_0(\omega)||\ )$$

Since $a(t_0,\cdot) \in \varkappa$ wee can find a $\delta = \delta(t_0,\in\eta) > 0$ such that
$P\{\omega : a(t_0, \| x_0(\omega)\|) > \delta_1\} = P\{\omega : \| x_0(\omega)\| > \delta\}$

Now, we claim that (SP_1) holds. Suppose that this claim is false . Then, there would exist a solution process $x(t,\omega)$ of (1) with $P\{\omega : \| x_0(\omega) \| > \delta\} < \eta$ and a $t_1 > t_0$ such that

(18) $P\{\omega : \| x(t_1, \omega) \geq \varepsilon\} = \eta$.

By theorem 1, the inquality

(19) $V(t,x(t,\omega),\omega) \leq r(t,t_0,V(t_0,x_0,\omega),\omega),\omega)$

is valid for all $t \geq t_0, \omega \in \Omega$. From (15) and (19) , we have

(20) $b(\| (t,\omega)\|) \leq V(t,x(t,\omega),\omega)$

$\leq r(t,t_0, V(t_0,y(t,t_0,x_0,\omega),\omega),\omega)$

The relations (16), (18) and (20) lead us to the contradiction

$= P\{\omega : \| x(t_1,\omega) \| \geq \varepsilon\}$

$= P\{\omega : b(\| x(t_1,\omega)\| \geq b(\varepsilon)\}$

$\leq P\{\omega : r(t,t_0,V(t_0, y(t,t_0,x_0,\omega),\omega),\omega) \geq b(\varepsilon)\} < \eta$,

thus proving the statement (i).

Now, we shall give the proof of the statement (ii). Assume that (CSP_2^*) holds. Then given $b(\varepsilon) > 0, \eta > 0$, and $t_0 \in R^+$, there exist positive numbers $\delta^0(t_0) = \delta^0$ and $T = T(t_0,\varepsilon,\eta)$ such that(21) $P\{\omega : r(t,t_0 V(t_0,y(t_0 y(t,t_0,x_0,\omega),\omega),\omega) \geq h(\varepsilon)\} < \eta, t \geq t_0+T$ whenever

$P\{\omega : \| x_0(\omega) \| > \delta^0\} < \eta$.

To prove statement (ii) , it is enough to prove that for any $\varepsilon > 0$, $\eta > 0$ and $t_0 \in R^+$ there exist positive numbers $\delta_0 = \delta_0(t_0)$ and $T = T(t_0,\varepsilon,\eta)$ such that $P\{\omega : \| x_0(\omega) \| > \delta_0\} < \eta$ implies

(22) $P\{\omega : \| x(t,\omega) \| \geq \varepsilon \} < \eta, \quad t \geq t_0 + T.$

We claim that (22) holds, otherwise, there exists a sequence $\{t_n\}, t_n \geq t_0 + T$, $t_n \to \infty$ as $n \to \infty$ such that for some solution process of (1) satisfying $P\{\omega : \| x_0(\omega) \| \, \delta_0\} < \eta$, it will satisfy the relation

(23) $P\{\omega : \| x(t_n, \omega) \| \geq \varepsilon \} = \eta, \quad t_n \geq t_0 + T.$

This together with (19) and (21) will establish the vaalidity of (22) . This completes the proof of the theorem.

Our next theorem establishes the properties of (1) in the mean.

Theorem 3 : Assume that the hypotheses of Theorem 1 hold and (15) *hold with* $b \in V\kappa$, $a \in C\kappa$ then

(i) (CSM_1^*) *implies* (SM_1)

(ii) (CSM_2) *implies* (SM_2).

Proof

Let us first prove the statement (i). Let $\varepsilon > 0, t_0 \in R^+$ be given. Assume that (CSM_1^*) holds. Then for given $b(\varepsilon) > 0$ and $t_0 \in R^+$, there exists a positive function $\delta_1 = \delta_1(t_0,\varepsilon)$ such that $E[\| x_0(\omega) \|] \leq \delta_1$ implies

(24) $E[r(t,t_0,V(t_0,y(t,t_0,x_0,\omega),\omega),\omega)] < b(\varepsilon), \quad t \geq t_0.$

we choose $r_0(\omega)$ such that $V(t_0,y(t,t_0,x_0,\omega),\omega) = r_0(\omega)$ and

$$E[\| x_0(\omega) \|] = a(t_0, E[\| x_0(\omega) \|]).$$

Since $a \in C\kappa$, we can find a $\delta = \delta(t_0,\varepsilon) > 0$ such that

$$E[\| x_0(\omega) \|] \leq \delta \text{ implies } a(t_0, E[\| x_0(\omega) \|]) < \delta_1 .$$

Now we claim that if $E\ [||x_0(\omega)||] \leq \delta$, then $E[||x(t,\omega)||] < \varepsilon$, $t \geq t_0$. Suppose that this is false. Then, there would exist a solution process $x(t,\omega)$ of (1) with $E\ [\ ||x_0)\omega)||] \leq \delta$ and a $t_1 > t_0$ such that

(25) $E\ [||\ x(t_1,(\omega)\ ||] = \epsilon$ and $E\ [||\ x(t,\omega)||\ ] \leq \epsilon, t \in [t_0,t_1]$.

By Theorem 1 and (15) we have the inequality (20), and by the convexity of b we have

(26) $b(E[\ ||x(t,\omega)\ ||]) \leq E\ [\ V(t,x(t,\omega),\omega)]$

$\leq E\ [\ r(t,t_0,V(t_0,y(t,t_0,x_0,\omega),\omega),\omega)]$

The relations (24), (25) and (26) lead to the contradiction

$b(\varepsilon) \leq E[V(t_1, x(t_1,\omega),\omega)\ ]$

$\leq E[r(t,t_0,V(t_0,y(t_1,t_0,x_0,\omega),\omega),\omega)] < b\ (\epsilon)$,

Providing (i). The proof of (ii) can be formulated similarily .

In concluding this section we note that, one can very easily formulate other definitions and theoremms on stability and boundedness as in [4,5,14] analogously with suitable modifications for the integrodifferental system (1). We omit the details.

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ICHTHYOFAUNA OF THE RIVER GANGA AT THE FOOTHILLS OF GARHWAL HIMALAYA

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ABSTRACT

The paper deals with the Ichthyofauna of the river Ganga at foothills of Garhwal Himalaya. A systematic list of 44 species is given with their present valid names and earlier names given by Day and Hamilton.

Key words and Phrases : Ichthyofauna, Ganga river, Systematic list.

INTRODUCTION

In 1822 Hamilton [5] listed the fish fauna of the river Ganga and its tributaries, no other work has been published before this, Day [4], Hora [7], Hora and Mukurjee [8], Jayaram [9], Mahajan [10], Pant [12] and Venkateshwarlu and Menon [15] have also published reports on the Ichthyofauna of the Ganga river system.

The fish fauna of this river in the uplands have been studied by Heckel [6], Malik [11], Sehgal et al.[13] Badola [1], Badola and Pant [2], Badola and Singh [3] and Singh et al.[14]. But the reports on the fish fauna of the Ganga river at the foothills of Garhwal Himalaya are scanty and limited. The paper deals with the Ichthyofauna of the river Ganga and its tributaries (Song and Suswa) from Lakshman Jhula to Kangri Village, Hardwar (Table).

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TABLE : SYSTEMATIC LIST OF FISHES

Present Scientific Name A	Name according to Day B	Name according to Hamilton C	Local Name D
	Phylum :	VERTEBRATA	
	Subphylum :	CRANIATA	
	Super class :	GNATHOSTOMATA	
	Series :	PISCES	
	Class :	TELEOSTOMI	
	Sub class :	ACTINOPTERYGII	
	Order :	CYPRINIFORMES	
	Sub order :	CYPRINOIDEI	
	Family :	CYPRINIDAE	
	Sub family :	RASBORINAE	
	Genus :	*Barilius* Hamilton	
Barilius bendelisis (Ham.)	*Barilius bendelisis*	*Cyprinus* (*Barilius*) *bendelisis*.	Bhola
Barilius bola (Ham.)	*Barilius bola*	*Cyprinus* (*Barilius*) *Bola*	Nayar Bhola
Barilius vagra (Ham.)	*Barilius vagra* *Barilius modestus*	*Cyprinus* (*Barilius*) *vagra*	persee
	Genus : *Danio*	Hamilton	
Danio (*Danio*) *devario* (Ham.)	*Danio devario*	*Cyprinus* (*Babdio*) *devario*	Patukari
Danio (*Brachydanio*) *rerio*	*Danio rerio*	*Cyprinus* (*Danio*) *rerio* *Cyprinus* (*Danio*) *chapalio*	
	Genus : *Esomus* :	Swainson	
Esomus danricus (Ham.)	*Nuria danrica*	*Cyprinus* (*Danio*) *danrica* *Cyprinus* (*Danio*) *sunita* *Cyprinus* (*danio*) *jogia*	Denč

Contd..

A	B	C	D
	Genus : *Rasbora* Bleeker		
Rasbora daniconius (Ham.)	*Rasbora daniconius*	*Cyprinus* (*Danio*) *daniconius* *Cyprinus* (*Danio*) *anjana*	Dendua
	Sub family : Cyprininae		
	Ganus : *Chagunio* Smith		
Chagunio chagunio (Ham.)	*Barbus chagunio*	*Cyprinus* (*Cyprinus*) *chagunio*	Gelhari
	Genus : *Crossocheilus* Van hasselt		
Crossocheilus latius latius (Ham.)	*Cirrhina latia*	*Cyprinus* (*Garra*) *latius* *Cyprinus* (*Garra*) *gohama* *Cyprinus* (*Garra*) *sada*	Pet phorni
	Genus : *Garra* Hamilton		
Garra gotyla gotyla (Gray)	*Discognathus lamta* impart	*Cyprinus lamta*	Siltoka
	Genus : *Labeo* Cuvier		
Labeo boga (Ham.)	*Labeo boga*	*Cyprinus* (*Bangana*) *boga*	Bhagan
Labeo dero (Ham.)	*Labeo microphthalmus* *Labeo diplostomus* *Labeo sindensis*	*Cyprinus* (*Bangana*) *dero*	Arangi
Labeo gonius (Ham.)	*Labeo gonius*	*Cyprinus* (*Cyprinus*) *gonius* *Cyprinus* (*Cyprinus*) *curchiu* *Cyprinus* (*Cyprinus*) *cursa*	 kursi Bata

Contd...

A	B	C	D
	Genus : *Puntius* Hamilton		
Puntius sarana sarana (Ham.)	*Barbus sarana* *Barbus chrysopoma* *Barbus pinnauratus*	*Cyprinus (Cyprinus) Saran*	Darahee
Puntius sophore (Ham.)	*Barbus sophore* *Barbus stigma* *Barbus puntio* *Barbus chrysopterus*	*Cyprinus (puntius) sophore* *Cyprinus (puntius) puntio*	Sidhari
Puntius ticto (Ham.)	*Barbus ticto*	*Cyprinus (puntius) ticto*	Sidhari
	Barbus punctatus		
	Genus : *Tor* Gray		
Tor putitora	*Barbus tor* (in part	*Cyprinus (Cyprinus) putitora*	Mahaseer
Tor tor (Ham.)	*Barbus tor* (in part) *Barbus hexasticus*	*Cyprinus (Cyprinus) tor*	Mahaseer
	Sub family : **Schizothoracinae** Heckel		
	Genus : *Schizothorax*Heckel		
Schizothorax plagiostomus	*Oreinus plagiostomus*		Dhibrua
Schizothorax sinuatus	*Oreinus sinuatus*		Maseen

Contd..

A	B	C	D
Genus : *Schizothoraichthys* Misra			
Schizothoraichthys progastus (mc Clelland)	*Schizothorax progastus*		Chonga
Family : Cobitidae			
Genus : *Botia* Gray			
Botia dario (Ham.)	*Botia dario*	*Cabitus geto*	Baghaua
Genus : *Nemacheilus* Van Hasselt			
Nemacheilus beavani (*Gunther*)	*Nemacheilus beavani*		Natwa
Nemacheilus botia (Ham.)	*Nemacheilus botia*	*Cobitus botia* *Cobitus bilturio*	Natwa
Nemacheilus montanus (Mc Clelland)	*Namacheilus denisoni*		Natwa
Nemacheilus rupicola (Mc Clelland)	*Nemacheilus rupicola*		Natwa
Nemacheilus savona	*Nemacheilus savona*	*Cobitus turio*	Natwa
Division : SILURI SUB order ; SILUROIDEI Family : Bagridae			
Genus : *Mystus* Gronow (emend. Scopoli)			
Mystus cavasius (Ham.)	*Macrones cavasius*	*Pimelodus cavasius*	Sutahawa Tengara
Mystus tengara (Ham.)	*Macrones tengara*	*Pimelodus tengara*	Tengana
Mystus seenghala (Sykes)	*Macrones seenghala*		Seenghara

Contd..

A	B	C	D
	Genus : *Rita* Bleeker		
Rita rita (Ham.)	*Rita buchanani*	*Pimelodus rita*	Hunna Rita
	Family: Sisoridae Genus : *Bagarius* Bleeker		
Bagarius bagarius	*Bagarius yarrelli*	*Pimelodus bagarius*	Gonch
	Genus : *Glyptothorax* Blyth		
Glytothorax conirostris (*Steind*)	*Glyptosternum conirostri* *Glyptosternum lonah* (in part)		Nau
Glyptothorax pectinopterusrus (Mc Clelland)	*Glyptosternum pectinopterum*		Nau
Glyptothorax madraspatanum	*Glyptosternum modestum*		Nau
	Family : Schilbeidae Genus : *Clupiscma* Swainson		
Clupisoma garua (Ham.)	*Pseudeutropius garua*	*Silurus* (*Callichrous*) *garua*	Baikari Karahi
	Order : BELONIFORMES Suborder: SCOMERBOSOCOIDEI Family : Belonidae Genus : *Xenentodon* Regan		
Xenentodon cancila (Ham.)	*Belone cancila*	*Esor cancila*	Kauva

Contd..

A	B	C	D
	Order : OPHIOCEPHALIFORMES (CHANNIFORMES) Family : **Ophiocephalidae (Channidae)** Genus : *Channa* Scopoli		
Channa marulius (Ham.)	*Ophiocephalus marulius*	*Ophiocephalus marulius*	Saur
Channa punctatus (Bloch)	*Ophiocephalus punctatus*	*Ophiocephalus lata*	Girai
Channa striatua (Bloch)	*Ophiocephalus striatus*	*Ophocephalus wrahl* *Ophiocephalus chena*	Saur
Channa gachua (Ham.)	*Ophiocephalusgachua*	*Ophiocephalus gachua*	Chanaga
	Order : PERCIFORMES Suborder: ANABANTOIDEI Family : **Anabantidae** Ganus : *Colisa* Cuvier et valenciennes		
Colisa fasciatus(Block et schn.)	*Trichogaster fasciatus*	*Trichopodus colisa* *Trichopodius bejeus* *Trichopodius cotra*	Khosti
	Order : MASTACEMBELIFORMES Family : **Mastacembelidae** Genus : *Mastacembelus* Schopoli		
Mastacembelus armatus (Lacepede)	*Mastacembelus armatus*	*Macrognathus armatus*	Baam
Mastacembelus pancalus (Ham.)	*Mastacembelus pancalus*	*Macrognathus pancalus*	Malga Pataya

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EXTENSIONS OF THE CARÍSTI-KIRD FIXED POINT THEOREM

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ABSTRACT

Two coincidence theorems using an elementary constructive proof are proved in this paper which include *inter alia* a recent substantial generalization of the Caristi-Kirk fixed point theorem due to Bollenbacher and Hicks [1]. Serveral examples provide an insight into the results and raise new questions.

AMS (MOS) subject classifications (1880): 47H10, 54H25.

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1. INTRODUCTION

The Caristi-Kirk fixed point theorem [2] generalizes a number of fixed point theorems and has various proofs and several applications (see, for instance, [1]-[3], [5], [6]-[8], [10] and the referencesof [8]). However, Caristi's proof, as well as several others, lack iterative construction of the fixed point. A recent result of Bollenbacher and Hicks [1] (see also Hicks [5]) has an elementary constructive proof and improves the Caristi-Kirk theorem substantially.

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In this paper two coincidence theorems (Theorems 1-2) for a pair of maps on an arbitrary set with values in a metric space are given and the result of [1], as well as of others, may be derived as corellaries. Indeed, apart from obtaining the fixed point theorem of Bollenbacher and Hicks [1] exactly as a special case of Theorem 1 (Cf. Remark 1), a variant of their theorem is also obtained as a corellary (Corollary 3) of Theorem 2.

Throughout this paper, let A be an arbitrary set, (X, d) a metric space and f, T maps on A with values in X. A point z in A is said to be a coincidence point of f and T if $fz = Tz$. If for a point $x_0 \in A$ there exists a sequence $\{x_n\}$ of points of A such that $fx_{n+1} = Tx_n$ $n=0,1,2....$, then

$$O(T,f;x_0) = \{fx_n : n = 1,2,.....\}$$

is said to be the orbit of (T,f) at x_0. We shall use $O(T,f;x_0)$ as a set and as a sequence as the situation demands. A subspace of X is said to be $(T,f;x_0)$-orbitally complete if the colsure of $O(T,f;x_0)$ is complete. Fvidently, the completeness of X implies the orbital completeness, and the space may be $(T,f;x_0)$-orbitally complete without being complete (see, for instance, [11, Ex. 2.4]). A subspace of X will be called (T,f)-orbitally complete f it is $(T,f;x_0)$-orbitally complete for every $x_0 \in A$. $G:X \rightarrow (0,\infty)$ is $(T,f;x_0)$-orbitally lower semi-continuous (l.s.c.) if $\{z_n\}$ is a sequence in $O(T,f;x_0)$ and $\lim z_n=p$ implies $G(p) \leq \liminf G(z_n)$. If $A=X$ and $f=I$ (identity map) then G is said to be T-orbitally l.s.c. at x_0; [1], [8,p.62]. If $A = X$ then: (i) T is said to be $(T,f;x_0)$-orbitally continuous [12] if the restriction of T on the closure of $O(T,f;x_0)$ is continuous; (ii) T and f are said to be weakly commuting [9] iff $d(Tfx, fTx) \leq d(fx, Tx)$ for every x in **X**; and (iii) T and f will be called weakly x_0-preorbitally commuting (w.x_0-p.c.) iff there exists a positive integer N such that $d(Tfx_n, fTx_n) \leq d(fx_n, Tx_n)$ for x_n (in $\geq N$) occurring in $O(T,f;x_0)$. Moreover, T and f will be called (T,f)-orbitally continuous (respectively weakly preorbitally commuting) if they are $(T,f;x_0)$-orbitally continuous (resp. w.x_0-p.c.) for ever $x_0 \in X$. We remark that commutativeity $\Rightarrow$ weak commutativety $\Rightarrow$ weak preorbital commutativity but not its vice-versa, (see Example 2, 5 below).

2. RESULTS

Consider the following condition for $\phi: X \rightarrow [0, \infty)$ and every $y \subset A$:

(CP) $\qquad d(fy, Ty) \leq \emptyset(fy) - \emptyset(Ty).$

Park [6, Prop. 7] using a non-constructive proof shows the existence of a coincidence point of T and f under this condition when A = X is complete, ϕ l.s.c. and f surjective. The Caristi-Kirk fixed point theorem [2] states that T satisfying (CP) with A = X complete, f = I and $\emptyset$ l.s.c. has a fixed point. In the following Coincidence Theorems 1-2, assume that there exists an orbit for (T,f).

THEOREM 1. Let T, f be maps on A with values in X and $\phi : X \rightarrow (0, \infty)$. Suppose there exists an $x_0 \in A$ such that (CP) for every $fy \in O(T,f;x_0)$, and f(A) is $(T,f;x_0)$-orbitally complete. Then:

(1.1) $\lim Tx_n = \lim fx_n = fz$ exists for some $z \in A$;

(1.2) $d(Tx_n, fx) \leq \emptyset(Tx_n)$;

(1.3) $Tz = fz$ iff $G(fz) = d(fz, Tz)$ is $(T, f; x_0)$-orbitally i.s.c.;

(1.4) $d(Tx_n, Tx_0) \le Ø(Tx_0)$ and $d(fz, Tx_0) \le Ø(Tx_0)$.

PROOF. Since

$$\begin{aligned} d(Tx_n, Tx_{n+1}) &= d(fx_{n+1}, Tx_{n+1}) \\ &\le Ø(fx_{n+1}) - Ø(Tx_{n+1}) \\ &= Ø(Tx_n) - Ø(Tx_{n+1}), \end{aligned}$$

that is

$$\sum_{k=0}^{n} d(Tx_k, Tx_{k+1}) \le Ø(Tx_0) - Ø(Tx_{n+1}) \le Ø(Tx_0)$$

$\{Tx_n\}$ and $\{fx_n\}$ are Cauchy sequences and both have the same limit. Call it p. Since $p \in f(A)$, there exists a z in A such that $fz = p$. This proves (1.1).

$$(*) \qquad 0 \le d(Tx_n, Tx_m) \le \sum_{k=n}^{m-1} d(Tx_k, Tx_{k+1}) \le Ø(Tx_n) - Ø(Tx_m) \le Ø(Tx_n).$$

Making $m \to$ gives (1.2).

The $(T, f; x_0)$-orbital l.s.-continuity of G implies

$$\begin{aligned} 0 \le d(fz, Tz) = G(fz) &\le \liminf G(fx_n) \\ &= \liminf d(fx_n, Tx_n) \\ &= \liminf d(fx_n, fx_{n+1}) = 0. \end{aligned}$$

So $fz = Tz$.

If $Tz = fz$ and $\{fx_n\} \subset O(T, f; x_0)$ with $fx_n \to fz$ then $G(fz) = d(fz, Tz) = 0 \le \liminf d(fx_n, fx_{n+1}) = \liminf d(fx_n, Tx_n) = \liminf G(fx_n)$. This completes the proof of (1.3).

In view of the ineuality (*),

$$d(Tx_0, Tx_n) \le Ø(Tx_0) - Ø(Tx_n) \le Ø(Tx_0).$$

Making $n \to \infty$ gives $d(Tx_0, fz) \leq \phi(Tx_0)$. This ends the proof.

THEOREM 2. Let T, f be self-maps of X and $\phi: X \to [0, \infty)$. Suppose there exists an $x_0 \in X$ such that (CP) with A = X holds for every $fy \in O(T,f;x_0)$, and X is $(T,f;x_0)$-orbitally complete. Then :

(2.1) $\lim Tx_n = \lim fx_n = p$ exists, $p \in X$;

(2.2) $d(Tx_n, p) \leq \phi(Tx_n)$;

(2.3) if $\{T,f\}$ is a weakly x_0-preorbitally continuous pair and T, f are $(T,f;x_0)$-orbitally continuous then $T_p = fp$.

(2.4) $d(Tx_n, Tx_0) \leq \phi(Tx_0)$ and $d(p,Tx_0) \leq \phi(Tx_0)$.

PROORF. In view of the proof of Theorem 1 we need show only (2.3). Since $fx_n \to p$, $Tx_n \to p$, the w.x_0-p. commutativity of T and f implies $\lim d(Tfx_n, fTx_n) = 0$. This together with the orbital continuity of T and f gives $d(Tp, fp) = 0$, i.e. $Tp = fp$.

REMARKS 1. Theorem 3 of Bollenbacher and Hicks [1] generalizing the Caristi-Kirk fixed point theorem is obtained from Theorem 1 when A = X and f = I.

2. If $T(A) \subset f(A)$ then for any $x \in A$ there exists a $y \in A$ such that $fy=Tx$. Therefore $T(A) \subset f(A)$ implies the existence of an orbit $O(T,f;x_0)$ for every $x_0 \in A$. If A = X and T = I then $O(I,f;x_0) = \{x_n = fx_{n+1}: n = 0,1,2,\ldots\}$.

3. If, instead of f(A), T(A) is assumed $(T,f;x_0)$-orbitally complete in Theorem 1 then, as evident from the proof of Theorem 1, fz and Tz are just interchanged in (1.1) - (1.4).

In view of Remark 2, we have the following corollaries from Theorems 1-2.

COROLLARY 1. Let T, f be maps on A with values in X such that $T(A) \subset f(A)$, $f(A)$ a (T,f)-orbitally complete subspace of X and (CP) for everyy $y \in A$. Then, for any $x_0 \in A$, there exists a sequence $\{x_n\}$ in A such that $fx_{n+1} = Tx_n$, $n = 0, 1, 2, \ldots$, and the conclusions (1.1) - (1.4) are ture.

COROLLARY 2. Let X be a metric space and T, f self-maps of X such that $T(X) \subset f(X)$, X is (T, f)-orbitally complete and (CP) with $A = X$ holds for ever $y \in X$. Then, for any $x_0 \in X$, there exists a sequence $\{x_n\}$ in X such that $fx_{n+1} = Tx_n$, $n = 0, 1, 2, \ldots$, and the conclusions (2.1) - (2.2), (2.4) and the following are true :

(2.3′) if {T, f} is a weakly preorbitally commuting pair and T, f are (T,f)-orbitally continuous then Tp=fp.

Taking T = I in Theorem 2 we have the following fixed point theorem.

COROLLARY 3. Let $f: X \to X$ and $\phi : X \to [0,\infty)$. Suppose there exists an orbit $O(I,f;x_0)$ for some $x_0 \in X$ such that

(P) $\quad d(fy, y) \leq \phi(fy) - \phi(y)$

for ever $y \in O(I,f;x_0)$, and X is $(I,f;x_0)$-orbitally complete Then :

(2.1c) $\lim x_n = p$ exists, $p \in X$;

(2.2c) $d(x_n,p) \leq \phi(x_n)$;

(2.3c) $fp = p$ if f is $(I,f;x_0)$-orbitally continuous, that is p is a fixed point of f if it is continuous at p;

(2.4c) $d(x_n, x_0) \leq \phi(x_0)$ and $d(p,x_0) \leq \phi(x_0)$.

REMARK 4. Corollary 3 is a variant of [1, Th. 3] and an improved version of a result of Park [6, Prop.9]. It is noted in [6,p. 149] that the condition (P) includes a number of contractive type conditions for maps on X. Examples (see below) show that the range of applicability of our results is wider than *inter alia* Goebel's coincidence theorem (Theorem 3 below). He proved it using the Banach contraction principle, while, as also noted in [1, Remark 2], our results include many generalizaitons of Banach's fixed point theorem.

THEOREM 3[4]. Let T, f be maps on A with values in X such that $T(A) \subset f(A)$ and

(3.1) $\quad d(Tx, Ty) \leq q\, d(fx, fy)$

for all $x, y \in A$ and some $q \in (0, 1)$. If f(A) is a complete subspace of X then T and f have a coincidence point in A.

3. EXAMPLES

EXAMPLE 1. This example illustrates Theorem 1 and shows that Theorem 3 is not applicable. Let $A = \{a,b,c\}$; $X = \{1,2,3\}$; $Ta = Tb = 1$, $Tc = 2$; $fa = 1$, $fb = 2$, $fc = 3$; $\emptyset(1) = 5/2$, $\emptyset(2) = 4$, $\emptyset(3) = 6$; and d a metric on X such that $d(1, 2) = d(1,3) = 3/2$, $d(2, 3) = 2$. Then evidently $T(A) \subset f(A)$, $d(fx, Tx) \leq \emptyset(fx) - \emptyset(Tx)$ for every x in A, and a is a coincidence point of T and f. However, Theorem 3 is not applicalbe as T and f do not satisfy (3.1), since $d(Ta, Tc) = d(fa, fc)$.

EXAMPLE 2. Let X be the set of non-negative rationals, d the absolute value metric on X and T, f; $X \to X$ such that

$$Tx = \begin{cases} 2x^2, & x \leq 4 \\ x, & x > 4 \end{cases} \quad \text{and} \quad fx = \begin{cases} 4x, & x \leq 1 \\ 8x, & x > 1. \end{cases} \quad \text{Let}$$

$$\emptyset(x) = \begin{cases} 1 + x, & x \leq 4 \\ 2 + x, & x > 4. \end{cases}$$

Then $T(X) \not\subset f(x)$ and $d(fx, Tx) \leq \phi(fx) - \phi(TX)$ for every x in X. Moreover, for $x_0 = 1$,

$$x_n = 2^{-2^n+1}, \; n = 1, 2, 3, \ldots, \text{ and}$$

$O(T,f;x_0) = \{2^{-2^n+3} : n = 1, 2, 3, \ldots\}$ converges to 0.

Evidently, Theorem 1 applies. Theorem 2 also applies as it is easy to check that $\{T,f\}$ is a w.x_0-p.c. pair. It may be noted that T and f are not weakly commuting since $d(Tfx,fTx) > d(fx,Tx)$ for $\frac{1}{2} \leq x \leq 1$.

EXAMPLE 3. Let (X, d) be as in Example 2, $\emptyset(x)=x$, $T = I$ and

$$fx = \begin{cases} 5x, & x \leq 1 \\ 1 + x/2, & x > 1. \end{cases}$$

Since, at $y = 4$, $d(fy, y) = 1$ and $\emptyset(fy) - \emptyset(y) = -1$, the condition (P) of Corollary 3 does not hold for all $y \in X$. However, since

$$O(I,f;1/5) = \{fx_n : x_n = 1/5^{n+1}, \; n = 1,2 \ldots\}$$

and $$d(fx_n, x_n) = 5^{-n} - 5^{-n-1} = \emptyset(fx_n) - \emptyset(x_n),$$

the condition (P) is satisfied for the orbit $O(I,f;1/5)$. So Corollary 3 applies to f, and $O(I,f;1/5)$ converges to 0, a fixed point of f. Indeed, the conclusions (2.1c) - (2.4c) are true for $x_0 = 1/5$. For $x_0 = 4$ we can have the following two orbits :

$$O_1 := \{4/5^n : n = 1, 2, 3 \ldots\}$$

and $$O_2 := \{2^n + 2 : n = 1, 2, 3, \ldots\}.$$

It is easy to check that (P) is satisfied for O_1 and not for O_2. We land no where with O_2 while O_1 converges to 0, a fixed point of f.

EXAMPLE 4. This example shows that the conditions of Theorem 1 do not guarantee the existence of a common fixed point of maps T and f on X (=A). Let $X=\{x: x \geq 1/2\}$; $Tx = 2x$, $fx = 3x - 1/2$ and $\phi(x) = x$. Then $d(fx, Tx) = x-1/2 = \phi(fx) - \phi(Tx)$ for every x in X, and

$$O(T,f;1) = \{fx_n = 1 + (2/3)^{n-1} : x_n=1/2 + 2^{n-1}/3^n, \; n=1,2,....\} \to 1.$$

Evidenty $T\frac{1}{2} = f\frac{1}{2} = 1$, and Theorem 1 applies to the maps of this example, and the coincidence point ½ is not a common fixed point of T and f. Indeed T and f are fixed point free maps.

EXAMPLE 5. This example shows that the conditions of Theorem 2 do not guarantee the existence of a common fixed point of T and f. Let $X = \{x : x \geq 1\}$, $Tx = x + 7/x$, $fx = (7x + 15)/8$, d the absolute value metric on X and

$$\phi(x) = \begin{cases} 22/x, & x = 1 \\ x, & x \neq 1. \end{cases}$$

Then $T(X) = [8, \infty) \subset [11/4, \infty) = f(X)$, and for $x_0 = 1$,

$$O(T,f;x_0) = \{fx_n=8: x_n = 7, n = 1, 2, 3, ..\} \to 8.$$

Moreover, as $d(fx_n, Tx_n) = 0 = \phi(fx_n) - \phi(Tx_n)$, the (CP) of Theorem 2 holds for the orbit O(T,f;1). To check the weak 1-preorbital commutativity, we note that T and f are commuting at $x_n(=7)$, n = 1, 2, ... Thus Theorem 2 applies to the maps T and f, and 8 is a coincidence point of T and f. Evidently T and f have no common fixed points. Indeed, T is a fixed point free map. Note that T and f are not weakly commuting.

Recall that, as shown by Examples 4-5, the conditions of Theorems 1-2 are not sufficient to ensure the existence of a common fixed point of maps T, f on A = X. So we have the following :

QUESTIONS 1. Under what additional condition(s), maps T, f on A=X satisfying the conditions of Corollary 1/Theorem 1/Theorem 2 will have a common fixed point ?

2. Will corollary 2 yeild a common fixed point when T and f are continuous or commuting (possibly both) ?

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FLOW OF BLOOD THROUGH A LOCALLY CONSTRICTED TUBE - A MATHEMATICAL MODEL AND ITS STATISTICAL ANALYSIS

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ABSTRACT

A knowledge of the wall shear stress distribution is very important for locating the arterial diseases and finding out mean to cure or alternatively to reduce the drag or wall shearing stress in cases of severe stenosis. Therefore in this paper an attempt has been made to study the flow behaviour of the particulate suspension through a locally constricted tube and to obtain the wall shear stress distribution for different values of time, axial positions and number densities. A statistical analysis has been carried out in order to achieve the uncovering of the embedded relations between the various parameters in the study with the help of multiple regression and partial correlations.

Key words : Stenosis, Red Blood Cells, Shear Stress
Subject Classification : Blood Rheology 92019

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1. INTRODUCTION :

Interest in problem of mechanics of system with more than one phase has developed repidly in recent years. Frequently we are concerned with the motion of a liquid or gas, which contains a distribution of solid spherical particles. Such situation occur, for example, in the movement of dust-laden air, in problem offluidization, in environmental pollution in the process by which rain drops are formed by the coalescence of small droplets which might be considered as solid particles for the purpose of examining their movement prior to coalescence, combustion and more recently in blood flow studies.

Saffman [13] described the effect of dust by two parameters, namely the concerntration and relaxation time. These parameters measure the rate at which the velocity of a dust particle adjusts to change the fluid velocity and depends upon the size of individual particles. Michael and Miller [9], Liu [8] have studied the flow produced by the motion of an infinite plate in a dusty gas occupying semi-infinite spce above it. Rao [12], Nath [10], Verma and Mathur [14] have studied the flow of dusty viscous liquid in circular channels. Healy and Young [3] have obtained exact solution for the problem using Laplace transform technique. Gupta and Gupta [2] studied the flow of dusty gases through a channel with arbitrary time varying pressure gradient. Charya [1] studied the flow of dusty fluid through a constricted channel.

Recently, the attention of the researchers has also focussed on the study of flow behaviour of the suspension of particles through a locally constrticted tube in reference to an important disease termed as "stenosis" by medical scientists. Blood flowing through a stenotic vein develops greater resistance and drag on the tube walls and at times it happens to be fatal. Saffman's [13] dusty fluid serves as a better model to describe the blood as a binary system. Nayfeh [11] studied the pulsatile flow of liquid containing small solid particles through a long narrow circular tube. A knowledge of the wall shear stress distribution is very important for locating the arterial diseases and finding out means to cure or alternatively to reduce the drag or wall shearing stress in cases of severe stenosios. Therefore, in this paper an attempt has been made to study the flow behaviour of the particulate suspensions through a locally constricted tube and obtained wall shear stress distribution, through the study under a mathematical model. The model has been used to construct synthetic data, on an electronic computer using the numerical inversion formula of Laplace transform. This data has been statistically studied to bring forward the implications of the model. More importantly, the statistical validation of the model could be done only with a reliable real-life data, which the authors very much missed to have had.

2. FORMULATION OF THE MATHEMATICAL MODEL :

In a circular tube, we consider the origin on the axis of the tube and the x-axis along the axis of the tube. The motion the dusty viscous, incompressible fluid is considered along the axis of the tube under the influence of the time (t) - dependent pressure gradient. The boundary conditions for the fluid are same as those for classical fluid and no particle boundary conditions are required. For the present, the initial conditions are at $t \leq 0$.

$$u = v = \frac{\partial u}{\partial t} = 0$$

The appropriate momentum equations in cylindrical polar coordinates are :

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + v\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right) + \frac{KNo}{\rho}(v-u) \quad \dots (1)$$

$$m\frac{\partial v}{\partial t} = K(u-v) \quad \dots (2)$$

Where r is radial distance, u and v are velocities of fluid and dust particles respectively, K is the Stoke's resistance coefficient which, for spherical particles of radius r is $6\pi\mu r$. Also, ρ and u are the density and viscosity of the blood respectively, and No is the number density of red blood cells, p is the pressure. Radius of constricted region may be expressed as [11]:

$$R = a - \alpha_o\left(1 + \cos\frac{\pi z}{z_o}\right) \quad \dots (3)$$

where 'a' is the radius of non-constricted region, α_o is constant depending on lumen of tube, z_o is the axial distance and $2z_o$ is constricted length.

Introducing the following nondimensional quantities

$$r^* = \frac{r}{a},\ x^* = \frac{x}{a},\ p^* = \frac{pa^2}{\rho\nu_2},\ t^* = \frac{t\nu}{a^2},\ u^* = \frac{ua}{\nu},\ v^* = \frac{va}{\nu} \quad \dots (4)$$

In the equation (1) and (2) (droping the stars) we get

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COINCIDENCE THEOREMS IN LINEAR TOPOLOGICAL SPACES

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ABSTRACT

The notion of diminishing joint orbital diametral sum for a pair of commuting mappiing has been generalized to a Hausdorff locally convex space whose topology is generated by a family of seminorms. Subsequently, this notion is utilized to prove some coincidence theorems which generalize several known results.

AMS(MOS) Subject classifications(1980): 47H10, 54H25.

Key words and phrases : Coincidence theorems, locally convex linear topological spaces, diminishing joint orbital diametral sum.

1. INTRODUCTION

Several fixed point theorems for nonexpansive mappings in locally convex linear topological spaces are known. In this setting the notions of diminishing orbital diameters and normal structure were given by Tan [7]. These notions were the generalizations of the corresponding notions given earlier by Belluce and Kirk [1] and Brodskii and Milman [2]. Recently the notions of diminishing joint orbital diameteral sum and joint normal structure in a metric space were introduced by Mishra [5] and several coincidence theorems for a pair of commuting mappings on metric and Banach spaces were proved. In this paper it is shown that

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most of these results hold for the same class of mappings on a bounded subset of a Hausdorff locally convex linear topological space.

Let (E, τ) be a locally convex linear topological space. A family $\{p_\alpha : \alpha \in \Lambda\}$ of seminorms on E is said to be an associated family of seminorms for τ if the family $B = \{tU : t > 0\}$ where $U = \bigcap_{i=1}^{n} U_{\alpha_i}$ and $U_{\alpha_i} = \{x \in E : p_{\alpha_i}(x) < 1\}$ forms a base of neighbourhoods of zero for τ. The set U is also given by $U = \{x \in E : p(x) < 1\}$ where p is the seminorm max $\{p_{\alpha_1}, p_{\alpha_2}, \ldots, p_{\alpha_n}\}$. A family $\{p_\alpha : \alpha \in \Lambda\}$ of seminorms on E is called an augmented associated family of seminorms for τ if it is an associated family for τ and has further property that given $\beta, \gamma \in \Lambda$ the seminorm max $\{p_\beta, p_\gamma\} \in \{p_\alpha : \alpha \in \Lambda\}$. We shall denote an associated family and an augmented associated family for τ by D and D* respectively. It is well-known that given a locally convex linear topological space (E, τ), there always exists a family of seminorms $\{p_\alpha : \alpha \in \Lambda\}$ such that $\{p_\alpha : \alpha\ \Lambda\} = D^*$. For details we refer to Köthe [4]. Unless otherwise stated, throughout this paper, E will denote a Hausdorff locally convex linear topological space (E, τ) defined by $\{p_\alpha : \alpha \in \ \} = D^*$.

DEFINITION 1.1 [9]. Let M be a nonempty subset of E. A mapping $f : M \to M$ is called D-nonexpansive (resp. D*-nonexpansive) if for all $x, y \in M$ and for each $P_\alpha \in D$ (resp. for each $p_\alpha \in D^*$), $p_\alpha(f(x) - f(y)) \leq p_\alpha(x-y)$.

Since f is D-nonexpansive $\Leftrightarrow$ f is D*-nonexpansive (cf. Tarafdar [8]), we shall simply say f is nonexpansive. For equivalent difinitions of nonexpansive mappings, we refer to Srivastava and Srivastava [6], Tan [7] and Taylor [10].

Let S and T be commuting mappings of E into E, and let $f = ST = TS$. For each $x \in E$, let $0(f^n(x)) = \{f^n(x), f^{n+1}(x), \ldots\}$ where $n = 0, 1, 2, \ldots$ and $f^0(x) = x$. For $x, y, \in E$ and $\alpha \in \Lambda$, define

$$P_\alpha(x,y) = p_\alpha(Tx-Sx) + p_\alpha(Ty-Sy) + p_\alpha(Tx-Sy) + p_\alpha(Ty-Sx) + p_\alpha(Tx-Ty).$$

It is clear that $P_\alpha(x,y) = P_\alpha(y,x)$. For each nonempty subset A of E, define $J_\alpha(A) = \sup\{P_\alpha(x,y) : x,y \in A\}$. Further, suppose that $R_\alpha(f^n(x)) = J_\alpha(0(f^n(x)))$ and $r_\alpha(x) = \lim_{n} R(f^n(x))$. Then we have the following:

DEFINITION 1.2. If for each $x \in E$, there is an $\alpha \in \Lambda$ such that $J_\alpha(0(x)) < \infty$ and $r_\alpha(x) < J_\alpha(0(x))$ whenever $J_\alpha(0(x)) > 0$, then mappings $S, T : E \to E$ will be said to have diminishing joint orbital diametral sum (dimjodis) on E.

DEFINITION 1.3.(i) Let A be a bounded subset of E. A point $a \in A$ will be called a joint nondiametral point of A (with respect to $S,T : E \to E$) if there is an $\alpha \in \Lambda$ such that $\sup\{P_\alpha(a,x) : x \in A\} < J_\alpha(A)$.

(ii) A bounded subset K of E will be said to have joint normal structure (with respect to $S, T : E \to E$) if for each convex subset H of K which contains more than one point, there is a joint nondiametrial point of H.

For equivalent definitions in metric spaces, we refer to [5].

Let K be a nonempty subset of E. A mapping $g: K \to K$ is called affine if $g(kx+(1-k)y) = k\,gx + (1-k)y$ for all $x, y \in K$ and for all $k \in [0,1]$.

PROPOSITION 1.4. Let $S, T : E \rightarrow E$, and let $A, B \subseteq E$ such that $A \subseteq B$. Then for each $\alpha \in \Lambda$,

(i) $J_\alpha(A) \le J_\alpha(B)$,

(ii) $J_\alpha(A) = J_\alpha(\bar{A})$, if S and T are continuous, where $\bar{A}$ is the colsure of A.

PROOF. The proof of (i) is obvious. We shall prove (ii). Let $\alpha \in \Lambda$ be arbitrary. Since $A \subseteq \bar{A}$ it follows that $J_\alpha(A) \le J_\alpha(\bar{A})$. To prove that $J_\alpha(\bar{A}) \le J_\alpha(A)$, let $x, y \in \bar{A}$, and let $\epsilon > 0$. Since S and T are continuous, we may choose u,v in A such that $p_\alpha(Tx-Tu) < \epsilon$, $p_\alpha(Ty-Tv) < \epsilon$, $p_\alpha(Sx-Su) < \epsilon$, and $p_\alpha(Sy-Sv) < \epsilon$. Hence using the definition of $p_\alpha(x,y)$ and triangle inequality, it can be easily verified that

$$\begin{aligned} P_\alpha(x,y) &\le P_\alpha(u,v) + 3\,(p_\alpha(Tx-Tu) + p_\alpha(Ty-Tv)) + 2(p_\alpha(Sx-Su) \\ &\quad + p_\alpha(Sy-Sv)) \\ &\le P_\alpha(u,v) + 10\,\epsilon. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, it follows that $P_\alpha(x,y) \le P_\alpha(u,v)$. Hence $J_\alpha(\bar{A}) \le J_\alpha(A)$, proving (ii).

PROPOSITION 1.5: Let A be a subset of E, and let $S, T: E \rightarrow E$. Then for each $\alpha \in \Lambda$,

(i) $J_\alpha(A) = J_\alpha(A)_c$, if S and T are affine,

(ii) $J_\alpha(A) = J_\alpha\overline{(A)_c}$, if S and T are affine and continuous; where $(A)_c$ and $\overline{(A)_c}$ are the convex hull and closed convex hull of A respectively.

PROOF. We shall prove (i) as (ii) follows from (i) and 1.4 (ii). Since $A \subseteq (A)_c$ we have from 1.4 (i) that $J_\alpha(A) \le J_\alpha(A)_c$. To prove that $J_\alpha(A)_c \le J_\alpha(A)$, let $x, y \in (A)_c$. Then we may assume that $\Lambda = \Sigma\, a_i x_i$ and $y = \Sigma\, b_j x_j$

and $y = b_j x_j$ where a_i, $b_j > 0$, $\sum a_i = 1 = \sum b_j$ and $x_i, x_j \in A$, $i=1,2, \ldots, n$ and $j = 1,2, \ldots, n$. Then following [5] it can be shown that

$$p_\alpha\,(Tx-Sx) \leq \Sigma\, b_j\, \Sigma a_i\, p_\alpha\,(Tx_i - Sx_i)\,,$$
$$p_\alpha\,(Ty-Sy) \leq \Sigma\, b_j\, \Sigma a_i\, p_\alpha\,(Tx_j - Sx_j)\,,$$
$$p_\alpha\,(Tx-Sy) \leq \Sigma\, b_j\, \Sigma a_i\, p_\alpha\,(Tx_i - Sx_j)\,,$$
$$p_\alpha\,(Ty-Sx) \leq \Sigma\, b_j \Sigma\, a_i\, p_\alpha\,(Tx_j - Sx_i),$$
$$p_\alpha\,(Tx-Ty) \leq \Sigma\, b_j \Sigma\, a_i\, p_\alpha\,(Tx_i - Tx_j)$$

for each $\alpha \in \Lambda$.

Therefore $P_\alpha(x,y) \leq (\Sigma\, b_j)\ (\Sigma\, a_i\, P_\alpha(x_i,x_j)) \leq (\Sigma b_j)\,(\Sigma\, a_i)\, J_\alpha\,(A)$. Hence $J_\alpha\,(A)_c \leq J_\alpha\,(A)$. This proves that $J_\alpha\,(A) = J_\alpha\,(A)_c$.

2. MAPPINGS WITH DIMJODIS

In this section we shall prove two coincidence theorems for mappings with diminishing joint orbital diametral sum (dimjodis) and with normal structure. The results obtained herein generalize certain theorems of Mishra [5, Theorems 3.2 and 3.3] and Tan [7].

DEFINITION 2.1. Let $a \in E$. Define $B_\alpha\,(a,r) = \{x \in E\ :\ P_\alpha(a,x) \leq r\}$, $r > 0$ and $\alpha \in \Lambda$.

THEOREM 2.2. Let $K \subseteq E$ be nonempty bounded closed convex, and let $M \subseteq K$ be nonempty weakly compact. If S and T are non-expansive commuting affine mappings of K into K such that

(i) $\overline{O(x)}_c \cap M \neq \phi$ for each $x \in K$, and
(ii) S and T have *dimjodis* on K,

then there exists a $z \in M$ such that $Sz = Tz$ i.e. S and T have a coincidence point z in M.

PROOF. Let $f = ST = TS$. A nonempty subset K^* of K minimal with respect to being closed convex f-invariant and having points in common with M is obtained by using weak compactness of M and the Zorn's lemma. Let $M^* = K^* \cap M$.

Let $x \in K^*$ and $\alpha \in \Lambda$. Then we wish to show that $J_\alpha(0(x)) = 0$. Suppose $J_\alpha(0(x)) > 0$. By (ii), there exists an integer N such that $k_\alpha = J_\alpha(0(f^n(x))) < J_\alpha(0(x))$ Suppose $L = \{z \in K^* : p_\alpha(z, f^N(x)) \leq k_\alpha$ for almost all n$\}$. Then $L \subseteq K^*$. We shall show that L is nonempty convex invariant under f and has points in common with M. Taking $f^m(x) \in 0(f^N(x))$, i.e. $m \geq N$ we have $p_\alpha(f^m(x), f^n(x)) \leq k_\alpha$ for all $n, m \geq N$. Hence $f^m(x) \in L$ and thus $L \neq \phi$. Suppose $y \in L$. Then for some intger N_1 we have $p_\alpha(y, f^n(x)) \leq k_\alpha$ for all $n \geq N_1$. Consider

$$\begin{aligned} p_\alpha(f(y), f^{n+1}(x)) &= p_\alpha(Tf(y) - Sf(y)) + p_\alpha(Tf^{n+1}(x) - Sf^{n+1}(x)) \\ &\quad + p_\alpha(Tf(y) - Sf^{n+1}(x)) + p_\alpha(Tf^{n+1}(x) - Sf(y)) \\ &\quad + p_\alpha(Tf(y) - Tf^{n+1}(x), \end{aligned}$$

Since f is nonexpansive (as S and T are so) and S and T are commuting, it follows that

$$\begin{aligned} p_\alpha(f(y), f^{n+1}(x)) &\leq p_\alpha(Ty - Sy) + p_\alpha(Tf^n(x) - Sf^n(x)) \\ &\quad + p_\alpha(Ty - Sf^n(x) + p_\alpha(Tf^n(x) - Sy) \\ &\quad + p_\alpha(Ty - Tf^n(x)) \\ &= p_\alpha(y, f^n(x) \leq k_\alpha \end{aligned}$$

for all $n + 1 \geq N_1 + 1$. Hence $f(y) \in L$ and thus f maps L inot L. It follows from the continuity of f that $f(\bar{L}) \sqsubseteq \overline{f(L)} \sqsubseteq \bar{L}$. Therefore f maps L into L. Also $0(f^N(x)) \sqsubseteq \bar{L}$, if follows that $\overline{0(f^N(x))_c} \sqsubseteq \bar{L}$. Therefore by (ii) we have $\bar{L} \cap M \neq \phi$. Hence the minimality of K* implies that $K^* = \bar{L}$.

Now let $u \in K^*$. Then since $u \in \bar{L}$, for $\in > 0$ there exists a point u* inL such that $p_\alpha(u-u^*) < \in/5$. Moreover, we can find an N_2 such that $p_\alpha(u^*, f^n(x)) \leq k_\alpha$ for all $n \geq N_2$. Further,

$$\begin{aligned} P_\alpha\ (u, f^n(x)) = {} & p_\alpha\ (Tu-Su) + p_\alpha(Tf^n(x)-Sf^n(x)) + p_\alpha\ (Tu-Sf^n(x)) \\ & + p_\alpha(Tf^n(x)-Su) + p_\alpha(Tu-Tf^n(x)) \\ \leq {} & p_\alpha(Tu-Tu^*) + p_\alpha(Tu^*-Su^*) + p_\alpha(Su^*-Su) \\ & + p_\alpha(Tf^n(x)-Sf^n(x)) + p_\alpha(Tu-Tu^*) \\ & + p_\alpha(Tu^*-Sf^n(x)) + p_\alpha\ (Tf^n(x) - Su^*) \\ & + p_\alpha(Su^*-Su) + p_\alpha(Tu-Tu^*) + p_\alpha(Tu^*-Tf^n(x)). \end{aligned}$$

No using the nonexpansiveness of S and T it can be easily verified that $P_\alpha\ (u, f^n(x)) \leq P_\alpha(u^*, f^n)) + 5\, p_\alpha(u-u^*) \leq k_\alpha + \in$ for all $n \geq N_2$.

Let $B_\alpha\ (u, k_\alpha + \in) = \{y \in E : P_\alpha(y,u) \leq k_\alpha + \in\}$. Then we observe that $0(f^n(x)) \subseteq B_\alpha\ (u, k_\alpha + \in)$ for all $n \geq N_2$. Since S and T are continuous and affine, $B_\alpha(u, t_\alpha + \in)$ is closed and convex and hence $\overline{0(f^n(x))_c} \sqsubseteq B_\alpha(u, k_\alpha + \in)$ for all $n \geq N_2$. Also by (i) $\overline{0(f^n(x))_c} \cap M^* \neq \phi$ for al $n \geq N_2$. Hence the weak compactness of M* ensures the existence of a point t in $\bigcap_{n=N_2}^{\infty} \overline{0(f^n(x))_c} \cap M^*$ and consequently $t \in B_\alpha(u, k_\alpha + \in)$. Since $\in > 0$ and u ($\in K^*$) are arbitrary, $t \in B_\alpha(u, k_\alpha)$

for all $u \in K^*$. Hence the set $A = \{z \in K^* : K^* \subseteq B_\alpha(u,k_\alpha)$ is nonempty. Since S and T are affine and continuous, A, $\subseteq K^*$ is closed and onvex.

We shall show that $f(A) \subseteq A$. Suppose this is not the case. Then $f(z) \notin A$ for some $z \in A$. Clearly $H = B_\alpha(f(z), k_\alpha) \cap K^*$ is closed and convex. Since $z \notin A$, we have $z \in K^* \subseteq B_\alpha(Z,k_\alpha)$. Also $f(z) \in K^*$. Hence $f(z) \in B_\alpha(z,k_\alpha)$. Therefore $z \in H$ and H is nonempty. Since $f(z) \notin A$, there exists $a \in K^*$ such that $y \notin B_\alpha(f(z),k_\alpha)$. This implies that $P_\alpha(f(z),y) > k_\alpha$. Hence H is a prop~~p~~er subset of K^*. Let $y^* \in K^*$. Then $f(y^*) \in K^*$. Also $y^* \in K^* \subseteq B_\alpha(z,k_\alpha)$, it follows that $P_\alpha(y^*,z) \leq k_\alpha$. It can be easily verified that $P_\alpha(f(z),f(y^*)) \leq P_\alpha(z,y^*) \leq k_\alpha$. Therefore $f(y^*) \in H$, proving $f(H) \subseteq H$. Moreover, since $z \in H$, $\overline{0(z)}_c \subseteq H$ and hence from (i) $H \cap M \neq \phi$. This shows that H is a nonempty closed convex proper subset of K^* which is invariant under f and has points in common with M, a contradiction to the minimality of K^*. Hence $f(A) \subseteq A$.

Let $v,w \in A$. Then $P_\alpha(v,w) \leq k_\alpha$. Consequntly, $J_\alpha(A) \leq k_\alpha = J_\alpha(0(f^N(x))) \leq J_\alpha(0(x)) < J_\alpha(K^*)$. Therefore A is a proper subset of K^*. Hence our assumption that $J_\alpha(0(x)) > 0$ is incorrect. Therefore $J_\alpha(0(x)) = 0$ and $Sx = Tx$.

THEOREM 2.3. Let $K \subseteq E$ be nonempty bounded closed convex, and let $M \subseteq K$ be nonempty weakly compact. If S and T are non-expansive commuting affine mappings of K into K such that for each $x \in K$,

(i) $\overline{(0(x))}_c \cap M \neq \phi$, and

(ii) $\overline{(0(x))}_c$ has joint normal structure,

then there exists a point $z \in M$ such that $f(x)=z$, where $f = ST = TS$.

PROOF. Define K* as in the proof of Theorem 2.2. We shall show that for each $x \in K^*$, $0(x)$ is a singleton. Suppose that there is an $x \in K^*$ such that $0(x)$ contains more than one point. Then by (ii), there exists a point $y \in \overline{(0(x))}_c$ such that y is a joint nondiametral point (with respect to S and T). That is there exists an $x \in \Lambda$ such that $\sup \{P_\alpha(y,z) : z \in \overline{(0(z))}_c\} = s_\alpha < J_\alpha\overline{(0(x))}_c) = J_\alpha(0(x))$. Let $A = \{z \in K^* : K^* \subseteq B_\alpha(z,s_\alpha)\}$. It can be shown as in the proof of Theorem 2.2 that A is closed convex, $A \cap M \neq \phi$ and $f(A) \subseteq A$. By the minimality of K* it follows that $A = K^*$. Hence $J_\alpha(K^*) = J_\alpha(A) \leq s_\alpha < J_\alpha(0(x)) \leq J_\alpha(K^*)$ which is absured. Hence for each $x \in K^*$, $0(x)$ is a singleton. Hence $f(x) = x$ for all $x \in K^*$.

3. BOUNDED MAPPINGS

In this section we shall discuss further properties of mappings with diminishing joint orbital diametral sum (dimjodis) by using the notion of bounded mappings. Such mappings were first studied by Kirk [3]. The term bounded mappings is due to Tan [7].

DEFINITION 3.1. Let $K \subseteq E$ be nonempty. A mapping $f : K \to K$ is said to be bounded if and only if for each $\alpha \in \Lambda$ there is a real number $c_\alpha > o$ such that $p_\alpha(f^k(x), f^k(y)) \leq c_\alpha p_\alpha(x-y)$ for all $x,y \in K$ and all $k = 1,2,\ldots$.

In the above condition if we set $f = ST = TS$, where S and T are self mappings of K and $c_\alpha = 1$, then f is nonexpansive and this happens when both S and T are nonexpansive.

PROPOSITION 3.2. Let $K \subseteq E$ be nonempty bounded, and let S and T be commuting continuous mappings of K into K such that $f = ST=TS$ is bounded. Then for each $\alpha \in \Lambda$ the function $x \to R_\alpha(x) = J_\alpha(0(x))$ is continuous.

PROOF. Let $\{x_\lambda : \lambda \in I\}$ be a net in K such that $x_\lambda \to x$ for some $x \in K$. Let I* be a directed subset (directed by $\leq$) of I. To prove that the given function is continuous we shall whow that $R_\alpha(x_\lambda) \to R_\alpha(x)$ for each $\alpha \in \Lambda$. Consider

$$\begin{aligned} |R_\alpha(x_\lambda) - R_\alpha(x)| = {} & |\sup \{ P_\alpha(xy) : x,y \in O(x_\lambda) \\ & -\sup \{P_\alpha(x,y) : x,y \in O(x) \,| \end{aligned}$$

Further,

$$\begin{aligned} & \sup \{P_\alpha(x,y) : x,y, \in O(x_\lambda)\} \\ & = \sup_{i,j \geq 0} \{P_\alpha(Tf^i(x_\lambda) - Sf^i(x_\lambda)) \\ & + p_\alpha(Tf^j(x_\lambda) - Sf^j(x_\lambda)) + p_\alpha(Tf^i(x_\lambda) - Sf^j(x_\lambda)) \\ & + p_\alpha(Tf^i(x_\lambda) - Sf^i(x_\lambda)) + p_\alpha(Tf^i(x_\lambda) - Tf^j(x_\lambda))\}, \end{aligned}$$

$$\begin{aligned} & \sup \{P_\alpha(x,y) : x,y \in O(x)\} \\ & = \sup_{i,j \geq 0} \{p_\alpha(Tf^i(x) - Sf^i(x)) + p_\alpha(Tf^j(x) - Sf^j(x)) \\ & + p_\alpha(Tf^i(x) - Sf^j(x)) + p_\alpha(Tf^j(x) - Sf^i(x)) \\ & + p_\alpha(Tf^i(x) - Tf^j(x))\}. \end{aligned}$$

Therefore,

$$| R_\alpha(x_\lambda) - R_\alpha(x)| \leq \sup_{i,j \geq 0} \{A_1 + A_2 + A_3 + A_4 + A_5\}$$

where

$$\begin{aligned} A_1 &= | p_\alpha(Tf^i(x_\lambda) - Sf^i(x_\lambda)) - p_\alpha(Tf^i(x) - Sf^i(x))|, \\ A_2 &= | p_\alpha(Tf^j(x_\lambda) - Sf^j(x_\lambda)) - p_\alpha(Tf^j(x)) - Sf^j(x))|, \\ A_3 &= |p_\alpha(Tf^i(x_\lambda) - Sf^j(x_\lambda)) - p_\alpha(Tf^i(x)) - Sf^j(x))|, \\ A_4 &= |p_\alpha(Tf^j(x_\lambda) - Sf^i(x_\lambda)) - p_\alpha(Tf^j(x) - Sf^i(x))|, \\ A_5 &= | p_\alpha(Tf^i(x_\lambda) - Tf^j(x_\lambda)) - p_\alpha(Tf^i(x) - Tf^j(x))|. \end{aligned}$$

Let $\epsilon > 0$ and set $\xi = \min\{\epsilon/_{10}, \epsilon/_{10c_\alpha}\}$. Since $x_\lambda \to x$ and S and T are continuous, there exists $\lambda_o \in I^*$ such that for all $\lambda \geq \lambda_o$ we have

$$(3.2.1) \quad \begin{aligned} &p_\alpha(x_\lambda - x) < \xi, \\ &p_\alpha(Tx_\lambda - Tx) < \xi, \\ &p_\alpha Tx_\lambda - Sx) < \xi. \end{aligned}$$

Further,

$$\begin{aligned} &p_\alpha(Tf^i(x_\lambda) - Sf^i(x_\lambda)) \\ &\leq p_\alpha(Tf^i(x_\lambda) - Tf^i(x)) + p_\alpha(Tf^i(x) - Sf^i(x)) \\ &\quad + p_\alpha(Sf^i(x) - Sf^i(x_\lambda)), \end{aligned}$$

and

$$\begin{aligned} &p_\alpha(Tf^i(x_\lambda) - Tf^i(x)) \\ &= p_\alpha(f^i(Tx_\lambda) - f^i(Tx)) \leq c_\alpha p_\alpha(Tx - Tx), \\ &p_\alpha(Sf^i(x_\lambda) - Sf^i(x)) \leq c_\alpha p_\alpha(Sx - Sx). \end{aligned}$$

Therefore,

$$(3.2.2) \quad \begin{aligned} &|p_\alpha(Tf^i(x_\lambda) - Sf^i(x_\lambda)) - p_\alpha(Tf^i(x) - Sf^i(x))| \\ &\leq c_\alpha\{p_\alpha(Tx_\lambda - Tx) + p_\alpha(Sx_\lambda - Sx)\}, \end{aligned}$$

..... Similiarly,

$$(3.2.3) \quad \begin{aligned} &|p_\alpha(Tx_\lambda - Sx_\lambda) - p_\alpha(Tx_\lambda - Sx)| \\ &\leq p_\alpha((Tx_\lambda - Tx) + p_\alpha(Sx_\lambda - Sx). \end{aligned}$$

Therefore from (3.2.1) through (3.2.3) it follows that $A_1 < \epsilon/5$ for all $i \geq 0$ and for all $\lambda > \lambda_o$. Similarly, $A_2 < \epsilon/5$ for all $i \geq 0$ and for all $\lambda > \lambda_o$. Using similar arguments, each of A_3, A_4 and A_5 can be made less than $\epsilon/5$ for all i, j, ≥ 0 and for all $\lambda > \lambda_o$. Consequently,

$$|R_\alpha(x_\lambda) - R_\alpha(x)| < \epsilon \text{ for all } \lambda \geq \lambda_0 .$$

Therefore $R_\alpha(x_\lambda) \to R_\alpha(x)$.

COROLLARY 3.3. Let $K \subseteq E$ be nonempty bounded, and S and T be commuting nonexpansive mappings of K into K. Then the function $x \to R_\alpha(x) = J_\alpha(0(x))$ is continuous for each $\alpha \in \Lambda$.

THEOREM 3.4. Let $K \subseteq E$ be nonempty bounded, let S and T be commuting continuous mappings of K into K such that (3.4.1) S and T have *dimjodis*.

(3.4.2) f (= ST=TS) is bounded,

(3.4.3) there is an $x \in K$ such that a subsequency of the sequence $\{f^n(x)\}$ converges to a point $z \in K$.

Then $\lim_n f^n(x) = z$ and z is a coincidence point of S ant T, i.e. $Sz = Tz$.

PROOF. Let $\{f^{n(s)}(x)\}$ be a subequence of $\{f^n(x)\}$ with its limit z ($z \in K$). Then since $R_\alpha(x)$ is continuous, for each $\alpha \in \Lambda$, it follows that $\lim_n R_\alpha(f^{n(s)}(x)) = R_\alpha(z)$. Also $\lim_n R_\alpha(f^n(x)) = r_\alpha(x)$ and $\{R_\alpha(f^{n(s)}(x))\}$ is a subsequence of $\{R_\alpha(f^n(x))\}$, if follows that

(3.4.4) $\lim R_\alpha(f^{n(s)}(x)) = r_\alpha(x) = R_\alpha(z)$.

Since $\lim_n f^{n(s)}(x) = z$ and S and T are continuous, for $\epsilon > 0$ there exists a positive integer m such that for all $s \geq m$,

$$p_\alpha(f^{n(s)}(x) - z) < \epsilon/10\, c_\alpha ,$$

$$p_\alpha(f^{n(s)}(Tx) - Tx) < \epsilon/10\, c_\alpha ,$$

$$p_\alpha(f^{n(s)}(S\ ) - Sx) < \epsilon/10\, c_\alpha ,$$

Since f is bounded, we have for all $t \geq 1$,

$$(3.4.5) \quad p_\alpha(f^{n(s)+t}(Tx) - f^t(Tz)) < \epsilon/10$$

$$p_\alpha(f^{n(s)+t}(Sx) - f^t(Sz) < \epsilon/10.$$

Let N be a positive integer. Then

$$(3.4.6) \quad R_\alpha(f^{n(s)+N}(x))$$

$$= \sup_{i,j \geq N} \{p_\alpha(Tf^{n(s)+i}(x) - Sf^{n(s)+i}(x)) + p_\alpha(Tf^{n(s)+j}(x) - Sf^{n(s)+j}(x) + p_\alpha(Tf^{n(s)+i}(x) - Sf^{n(s)+j}(x)) + p_\alpha(Tf^{n(s)+j}(x) - Sf^{n(s)+i}(x)) + p_\alpha(Tf^{n(s)+i}(x) - Tf^{n(s)+j}(x))\}.$$

Also,

$$p_\alpha(Tf^{n(s)+i}(x) - Sf^{n(s)+i}(x))$$

$$= p_\alpha(f^{n(s)+i}(Tx) - f^{n(s)+i}(Sx)$$

$$\leq p_\alpha(f^{n(s)+i}(Tx) - f^i(Tz)) + p_\alpha(f^i(Tz) - f^i(Sz)$$

$$p_\alpha(f^i(Sz) - f^{n(s)+i}(Sx))$$

$$< p_\alpha(f^i(Tz) - f^i(Sz) + \epsilon/5 = p_\alpha(Tf^i(z) - Sf^i(z) + \epsilon/5,$$

follows from (3.4.5).

Using a similar computation for other terms of(3.4

we have for $s \geq t$,

$$R_\alpha(f^{n(s)+N}(x)) < \sup_{i,j\geq N} \{p_\alpha(Tf^i(z) - Sf^i(z)) + p_\alpha(Tf^i(z) - Sf^j(z) + p_\alpha(Tf^i(z) - Sf^j(z)) + p_\alpha(Tf^j(z) - Sf^i(z)) + p_\alpha(Tf^i(z) - Tf^j(z))\} + \epsilon$$

$$= R_\alpha(F^N(z)) + \epsilon.$$

Hence $r_\alpha(x) = \operatorname{Lim}_N R_\alpha(f^{n(s)+N}(x)) \leq \lim_N R_\alpha(f^N(z))$

$= r_\alpha(z) + \epsilon$ for all positive integers N .

Since $\epsilon > 0$ is arbitnary, it follows that $r(x) \leq r_\alpha(z)$.

Hence from (3.4.4), we have

(3.4.7) $R_\alpha(z) = r_\alpha(x) \leq r_\alpha(z)$.

Supposr $R_\alpha(z) > 0$. Then since S and T have dimjodis, we have $r_\alpha(z) < R_\alpha(z)$, a contradiction to (3.4.7). Hence $R_\alpha(z) = 0$, proving $Sz = Tz$. Further, since $r_\alpha(x) = 0 = \lim_n r_\alpha(f^n(x))$, it follows that sequence $\{f^n(x)\}$ is a Cauchy sequence. Therefore $\lim_n f^n(x) = z$.

COROLLARY 3.5. Let $K \subseteq E$ be nonempty bounded, and let S and T be commuting nonexpansive mappings of K into K having dimjodis ; and if for some $x \in K$ a subsequence of the sequency $\{f^n(x)$, where $f = ST = TS$, has a limit z in K, then $\lim_n f^n(x) = z$ and $Sz = Tz$.

Since $K \subseteq E$ is sequentially compact implies K is bounded, we have the following:

COROLLARY 3.6. Let $K \subseteq E$ be nonempty sequentially compact, and let S and T be commuting continuous mappings of K into K such that

(3.6.1) S and T have *dimjodis*, and

(3.6.2) $f \doteq ST = TS$ is bounded.

Then for each x in K, some subsequence of the sequence $\{f^n(x)\}$ converges to a point z in K and $Sz = Tz$.

COROLLARY 3.7. Let $K \subseteq E$ be nonempty sequentially compact, and let S and T be commuting nonexpansive mappings of K into K having *dimjodis* . Then for each x in K some subsequence of the sequence $\{f^n(x)\}$ where $f = ST = TS$, converges to a point z in K and $Sz = Tz$.

Theorm 3.4 and its corollaries generalize the corresponding results of Tan [7] and Mishra [5].

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PLANT-PARASITIC AND SOIL NEMATODES INFESTING LEGUMINOUS CROPS IN RISHIKESH AND HARIDWAR REGIONS (U.P.)

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ABSTRACT

A survey of soil samples from the roots of Leguminous crops collected from ten localities in and around Rishikesh and Haridwar (U.P.) yielded 31 species of plant-parasitic and soil nematodes. These include mainly ectoparasites, predators and free-living forms, only two endoparasitic nematodes viz. *Rotylenchulus reniformis* and *Meloidogyne javanica* were recovered. *Helicotylenchus dihystra* and *Hopolaimus indus* are the most frequent nematods found in this region.

Key words : Leguminous crops, plant-parasitic and soil nematodes.

INTRODUCTION

Leguminous crops like *Phaseolus mungo*, *P. radiata*, *P. acontifolius*, *Cajanus cajan*, *Pisum sativum* etc. are widely grown in west Uttar Pradesh. Beside serving as fodder to animals these are also important, for Nitrogen fixation. Das [1] reported several nematodes associated with leguminous crops for the first time in India. Later on many nematodes including some endoparasites viz., *Heterodera avenae*, *H. cajani*, *H. trifolli*, *Meloidogyne javanica*, *M. incognita*, *Rotylenchulus reniformis*, *Radopholus similis* have been reported infesting leguminous crops in several parts of India by Seshadri & Shivkumar [11], Prasad *et al.*[9], [2], Koshy [5], Koshy and Sosamma [6], Janarthanan *et al.*[3], Sitaramiah *et al.* (12), Mulk & Jairajpuri [7,8], Kaushik & Bajaj [4], Reddy [10], A world list of Nematodes pathogens associated with chickpea, ground nut, pearl millect, pigeonpea and Sorghum was provided by Sharma [13]. But Rishikesh and Haridwar region has not yet been surveyed by any worker for the recovery of nematodes associated with leguminous crops. Therefore, an attempt has been made to recover all nematode associated with leguminous crops in this region.

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MATERIALS AND METHODS

The soil samples were processed by Cobb's Sieving and decantation techniques. Nematodes were isolated with the help of Baermann's funnel. The nematodes, thus recovered, were fixed with hot 4% formaline. Mounting of nematodes was done in anhydrous glycerine after slow dehydration. DeMan's formula was used for denoting body dimension. Ten localities were chosen to collect the soil samples around the roots of different leguminous crops.

RESULT AND DISCUSSION

A survey of soil samples during 1989-90 around the roots of different leguminous crops yielded as many as 31 species of plant-parasitic and soil nematodes (Table-1). All these nematodes belong to four orders of Nematoda viz., Tylenchida, Dorylaimida, Mononchida and Rhabditida. The nematodes belonging to Tylenchida are ecto-and endoparasites. Dorylaimida includes free-living as well as ecto-parasites. Mononchids and Rhabditids are exclusively predatory and saprophagous respectively. Some nematodes have been studied in detail and are as follows:

1. *Tylenchorhynchus vulgaris* Upadhyay, Swarup and Sethi, 1972:

(5 ♀♀) L = 0.62 - 0.78 mm; a = 24-26, b = 4.1 - 4.4, c = 19-21, v = 53-55, Spear = 22-23 µm; nervering=105-110 um; excretory pore=115-128 µm.

2. *Helicotylenchus dihystra* (Cobb, 1912) Sher, 1963

(7 ♀♀) L = 0.48-0.54 mm; a = 27-30; b = 3.5-4.0; c = 35-38; v = 48-50; Spear = 23-25 µm; excretory pore = 105-145 um.

3. *Hopolaimus indicus* Sher, 1963

(10 ♀♀) L = 0.90-1.1 mm; a = 23-25, b = 4.3-4.6; c = 21-24; v = 54-57; Spear = 33-37 um; Nerve ring = 90-105 um; excretory pore = 85-98 µm.

(5 ♂♂) L = 0.82-0.90 mm; a = 21-23; b = 4.4-4.6 c=24-28; Spicules=35-40 um; Bursa = 100-115 µm.

4. *Rotylenchus* sp.

(5 ♀♀) L=0.87-0.95 mm; a=30-37; a=30-33; b=6.0-6.5; c=22-24; v=65-67; Spear=27-30 um; Nerve ring= 95-110 um; excretory pore=117-120 µm.

5. *Longidorus* sp.

(5 ♀♀) L=3.6-4.5 mm; a=135-148; b=11-14; c=55-62; v=51-54; Odontostyle=73-75 µm; Odontophore= 56-65 µm., Guiding ring = 23-27 µm; pre rectum = 230-245 µm, Basal Bulb=53-60 x 20-22 µm,

6. *Xiphinema* sp.

(2 ♀♀) L=1.30-1.32 mm; a=44-45; b=9-10; c=17;
V=45-46; Odontostyle=65 µm; Odontophore=73 µm

7. *Actinolaimus* sp.

(10 ♀♀) L=1.75-1.89 mm; a=51-55; b=6.7-7.2; c=6-7;
V=52-55; Odontostyle=18-20 µm; Odontophore=
21-24 µm.

8. *Thornenema* sp.

(5 ♀♀) L=1.30-1.36 mm; a=35-38; b=4.4-4.6; c=5.3-5.6;
V=38-40; Odontostyle=11-13 µm; Odontophore=
15-20 µm.

9. *Mylonchulus minor* (Cobb, 1893) Andrassy, 1958;

(10 ♀♀) L=1.05-1.15 mm; a=23-25; b=3.2-3.5; c=30-32;
V=57-58; Buccal cavity = 29-35 x 14-17 µm;
tail = 34-35 µm.

10. *Iotonchus indicus* Jairajpuri, 1969;

(9 ♀♀) L=1.78-1.90 mm; a=31-35; b=4.4-4.7; c=8.1-8.6;
V=57-58; Buccal cavity = 43-47 x 23-25 µm.

From the present study it is clear that large number of nematodes are associated with the leguminous crops in this region. *Helicotylenchus dihystra, Hoplolaimus indicus, Tylenchorhynchus -vulgaris, Eudorylaimus* sp., *Longidorus* sp. and mylonechulus minor were most abundant while *Hirschmaniella, Aphelenchus, Scutellonema, Paratrichodorus* and *Rotylenchulus* were least encountered here.

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TABLE-1

LIST OF NEMATODES RECORDED FROM DIFFERENT LOCALITIES & DIFFERENT SAMPLES (Numbrs)

	NAMES OF NEMATODES	NUMBER OF SAMPLES
1.	*Longidorus* sp.	1, 2, 5, 6, 7, 9
2.	*Hoplolaimus indicus*	1, 2, 3, 4, 8, 9
3.	*Helicotylenchus dihystra*	1, 2, 3, 6, 7, 8, 9, 10
4.	*Rotylenchus* sp.	6, 8, 9
5.	*Aporcelaimus* sp.	1, 2
6.	*Hirschmanniella mucronata*	1
7.	*Pratylenchus* sp.	8
8.	*Rotylenchulus reniformis* sp.	3
9.	*Eudorylaimus* sp.	1, 2, 3, 4, 5, 7, 9
10.	*Dorylaimus stagnalis*	4, 5, 6, 7
11.	*Xiphinema* sp.	3
12.	*Prodorylaimus* sp	4, 6, 9
13.	*Tylenchorhynchus vulgaris*	1, 2, 3, 4, 6, 8, 10
14.	*Thornonema* sp.	3, 4, 6, 7, 8
15.	*Scutellonema* sp.	4, 2
16.	*Mononchus agnaticus*	4, 6
17.	*Amphidelus* sp.	5, 9
18.	*Mylonchulus minor*	5, 6, 7, 8, 9, 10
19.	*Discolaimus brevis*	4, 6, 9, 10
20.	*Aphelenchus avenae*	7
21.	*Para-phelenchus* sp.	7
22.	*Paratrichodorus mirzai*	7
23.	*Iotonchus indicus*	3, 7, 8, 10
24.	*Paratylenchus* sp.	8
25.	*Meloidogyne javanica*	8, 9
26.	*Aphelenchoides andrassyi*	8
27.	*Proleptonchus* sp.	9
28.	*Actinolaimus* sp.	2, 3, 5, 7, 8
29.	*Chronogaster* sp.	2, 3
30.	*Plectus* sp.	4
31.	*Rhabditis* sp.	8

Table-1 Contd.

LOCALITIES : SOIL AROUND THE ROOTS OF :

1. Pigeon Pea (*Cajanus cajan*) from Ram Nagar, Rishikesh, Dehradun.

2. Pigeon Pea (*Cajanus cajan*) from Geeta Nagar, Rishikesh, Dehradun.

3. Dhaicha (*Crotoloria junicea* L.) from Bhadra Bad, Haridwar.

4. Urd (*Phaseolus radiata*) from Sikenderpur, Haridwar.

5. Pigeon Pea (*Cajanus cajan*) from Dhanorie, Haridwar.

6. Bean Plant (*Dolichos lab-lab*) from Shyampur, Haridwar.

7. Chick Pea (*Cicer aretenum*) or Gram from Ganga Nagar, Rishikesh, Dehradun.

8. Matar (*Pisum sativum*) from I.D.P.L., Virbhadra, Rishikesh.

9. Chick Pea (*Cicer aretenum*) or Gram from Gomani Wala, Shyampur, Rishikesh, Dehradun.

10. Master (*Pisum sativum*) from Dhalwala, Rishikesh, Dehradun.

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बहुमानी संकारकों के अनुक्रमों का अभिसरण

श्याम लाल सिंह* उमेश चन्द्र गैरोला* एवं स्वामी नाथ मिश्रा**

(प्राप्त 6.12.1990)

सारांश

इस प्रपत्र में दो बहुमानी प्रतिचित्रणों के लिए इसिकावा पुनरावृत्तिक विधि के अधीन दो स्थिर बिन्दु प्रमेय स्थापित किये गये हैं, जो नैम्पल्ली-सिंह एवं सिंह व अन्य के परिणामों को विस्तारित करते हैं.

AMS (MOS) Subject Classifications (1980, Revision 1985). 54H25, 47H10.

ABSTRACT

CONVERGENCE OF SEQUENCES OF MULTIVALUED OPERATORS

S.L. Singh*, U.C. Guirola* and S.N. Mishra**

The notion of Ishikawa iterates is extended to a pair of multivalued maps and it is shown hat under certain contractive conditions the limit of the sequence of Ishikawa iterates, when it converges, is a common fixed point of the maps.

भूमिका एवं प्रारंभिकी

रोअडेस् [10], हिक्स-कुबिसेक [3], तथा अन्य कई गणितज्ञों (देखें, उदाहरणार्थ [9], [10], [12]) ने दिखाया है कि कोई प्रतिचित्रण T जो किसी विशेष संकुचित शर्त को सन्तुष्ट करता है उसके स्थिर बिन्दु पर वह मान पुनरावृत्तिक (देखें. [2], [14]) अनुक्रम अभिसरित होता है जो स्वयं अभिसारी हो, दूसरी ओर नैम्पल्ली - सिंह [9] ने व्यापक संकुचन शर्त के साथ यह देखा कि कोई अभिसारी इशिकावा [4] पुनरावृत्तिक अनुक्रम उसी बिन्दु पर अभिसरित होता है यदि बह बिन्दु प्रतिचित्रण T का स्थिर बिन्दु हो जब-कि नायडू-प्रसाद [8] ने [9] में प्रदत्त कुछ परिणामों का अध्ययन इशिकावा पुनरावृत्तिक अनुक्रम के अधीन एक युग्म प्रतिचित्रण के लिये किया हाल ही में कुहर्फिटिग [6] ने मान पुनरावृत्तिक विधि हेतु क्रेस्नोसेस्कोज [5] की पुनरावृत्तिक विधि का विस्तार बहुमानी प्रतिचित्रणों के लिये किया इन सभी धारणाओं से प्रेरणा लेकर सिंह [15] ने बहुमानी प्रतिचित्रणों के लिये इशिकावा पुनरावृत्तिक विधि को पारिभषित किया तथा इसके लिये स्थिर बिन्दुओं का सन्निकटन प्राप्त किया।

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प्रस्तुत प्रपत्र में एक युग्म बहुमानी प्रतिचित्रणों के लिए इशिकावा पुनरावृत्तिक विधि के अधीन दो प्रमेयों को सिद्ध कर रहे हैं. प्राप्त परिणाम सिंह [15] का व्यापकीकरण करते हैं. साथ ही रोअडेस् [10], हिक्स - कुबिसेक [3], नैम्पल्ली - सिंह [9], नायडू-प्रसाद [8] तथा कुहाफिटिग [6] द्वारा एक मानी एवं बहुमानी प्रतिचित्रणों के लिए स्थापित परिणाम विशिष्ट दशाओं के साथ समुचित रूप से प्राप्त किये जा सकते हैं.

सामान्यतया नाडलर [7] एवं सिंह [15] में प्रयुक्त संकेताक्षरों का अनुसरण किया गया है.

मान लें $(X,\|\ \|)$ एक मानकित रैखिक समष्टि है $CL(X)$ समष्टि X के अरिक्त संवृत उपसमुच्चयों का संग्रह है तथा $(CL(X),H)$ व्यापकीकृत हाउसडार्फ दूरीक समष्टि है.

निम्न प्रमेयिका सुज्ञात है (अधिक जानकारी के लिए देखें रूस [13]).

प्रमेयिका. मान लें $q>1$ एवं $A,B\in CL(X)$. तब A के प्रत्येक अवयव x के लिए B में एक बिन्दु y का अस्तित्व इस प्रकार प्राप्त होता है कि

$$d(x,y)\leq qH(A,B).$$

हम X के संवृत अवमुख उपसमुच्चय C से $CL(C)$ पर पारिभाषित प्रतिचित्रणों S,T के लिए [15] में चर्चित निम्न पुनरावृत्तिक विधि का अनुसरण करेंगें.

(1) $x_o\in C$;

(2) $y_n=b_n p_n+(1-b_n)x_n, n\geq 0, p_n\in Sx_n,$

(3) $x_{n+1}=(1-a_n)x_n+a_n q_n, n\geq 0;$

उक्त प्रमेयिका के अनुसार $q_n\in Ty_n$ इस प्रकार है कि

(4) $\|p_n-q_n\|=k^{-h}H(Sx_n, Ty_n)$, जहाँ $h,k\in(0,1)$;

(5) सभी n के लिए $0\leq a_n, b_n\leq 1$;

(6) सीमा $a_n\geq 0$.

ध्यान देने योग्य है कि यदि $T:C\rightarrow C$ तो शर्त (4) आसानी से सन्तुष्ट हो जाती है क्योंकि $k^{-h}>1$.

परिणाम

प्रमेय 1. मान लें C मानकित रैखिक समष्टि X का संवृत अवमुख उपसमुच्चय है तथा $S,T:C\rightarrow CL(C)$ पुनः मान लें कि पूर्व पारिभाषित अनुक्रम $\{x_n\}$ बिन्दु z पर अभिसरित होता है. यदि C के सभी अवयवों x,y के लिए S,T शर्त

(7) $$H(Sx, Ty) \le \text{अधिकतम} \{ ||x-y||, D(x,Sx), D(y,Ty), [D(y,Sx)+D(x,Ty)] \}$$

को संतुष्ट करे, तब $z \in Sz \cap Tz$.

उपपत्ति . क्योंकि $x_n \to z$ और $\{a_n\}$ शून्य से उपरि परिबद्ध है, अतः

$$||x_{n+1}-x_n|| = a_n||x_n-q_n||$$

एवं

$$||q_n-z|| \le ||z-x_n|| + ||x_n-q_n||$$

अर्थात् $||x_n-q_n||$ एवं $||q_n-z||$ शून्य पर अभिसरित होते हैं. पुनः त्रिभुज असमिका एवं (2) से

$$||x_n-y_n|| = b_n||x_n-p_n|| \le ||x_n-p_n||$$

$$||x_n-p_n|| \le ||x_n-q_n|| + ||q_n-p_n||$$

$$\begin{aligned} ||y_n-q_n|| &\le ||y_n-x_n|| + ||x_n-q_n|| \\ &\le ||x_n-p_n|| + ||x_n-q_n|| \\ &\le ||x_n-q_n|| + ||p_n-q_n|| + ||x_n-q_n|| \\ &\le 2||x_n-q_n|| + ||p_n-q_n|| \end{aligned}$$

एवं

$$||y_n-p_n|| = (1-b_n)\, ||x_n-p_n|| \le ||x_n-p_n||.$$

अतः

$$||p_n-q_n|| \le k^{-h}H(Sx_n,Ty_n)$$

$$\le k^{-h} \text{ अधिकतम } \{ ||x_n-y_n||, D(x_n,Sx_n), D(y_n,Ty_n), [D(x_n,Ty_n)+D(y_n,Sx_n)] \}$$

$\leq k'$ अधिकतम $\{ ||x_n-p_n||, ||y_n-q_n||, ||x_n-q_n|| + ||y_n-p_n||],$

जहाँ $k^{1-h} = k'$. इसलिए

$$||p_n-q_n|| \leq \frac{2k'}{1-k'} ||x_n-q_n||.$$

इस प्रकार $n \to \infty$ लेने पर

$$||p_n-q_n|| \to 0, \quad ||x_n-p_n|| \to 0,$$

तथा

$$||p_n-z|| \to 0, \quad ||y_n-q_n|| \to 0$$

एवं (2) से $||y_n-z|| \to 0$.

शर्त (7) द्वारा,

$$\begin{aligned}
D(z,Sz) &\leq ||z-q_n||+D(q_n,Sz) \\
&\leq ||z-q_n||+H(Ty_n,Sz) \\
&\leq ||z-q_n||+k \text{ अधिकतम}\{ ||y_n-z||, D(y_n,Ty_n), \\
&\qquad D(z,Sz), [D(y_n,Sz)+D(z,Ty_n)\} \\
&\leq ||z-q_n||+k \text{ अधिकतम}\{ ||y_n-z||, ||y_n-q_n||, \\
&\qquad D(z,Sz), ||y_n-z|| + D(z,Sz) + ||z-q_n||]\}.
\end{aligned}$$

स्पष्टतः $n \to 0$ लेने पर, $z \in Sz$. इसी तरह $z \in Tz$.

अब प्रतिचित्रणों $S,T:C \rightarrow CL(C)$ के लिए निम्न शर्तों पर विचार करें.

(8a) $H(x,Sx)+H(y,Ty) \le a||x-y||,\ 1 \le a < 2;$

(8b) $H(x,Sx) + H(y,Ty)$

$\le b[D(x,Ty) + D(y,Sx) + ||x-y||]$, जहाँ $1/2 \le b < 2/3$,

(8c) $H(x,Sx) + H(y,Ty) + H(Sx,Ty)$

$\le c[D(x,Ty) + D(y,Sx)],\ 1 \le c < 3/2,$

(8d) $H(Sx,Ty) \le k$ अधिकतम $\{||x-y||, D(x,Sx),$

$D(y,Ty),\ 1/2[D(x,Ty) + D(y,Sx))\},\ 0 \le k < 1$

प्रमेय 2. मान लें C मानकित रैखिक समष्टि X का एक संवृत अवमुख उपसमुच्चय है एवं पूर्व पारिभाषित अनुक्रम $\{x_n\}$ किसी बिन्दु $z \in C$ पर अभिसरित होता है. यदि C के प्रत्येक अवयव x,y के लिए प्रतिचित्रण S,T उक्त शर्तों (8a)-(8d) में से कम से कम एक शर्त को सन्तुष्ट करें तथा (8e) भी सन्तुष्ट हो तब $z \in S_z \cap T_z$, जहाँ

(8e) $k^{-h} \le 2$ या सीमा $b_n > \frac{(c-1-k^h)}{(c+1)}$.

उपपत्ति. हम किरिक [1] के निम्न सरल कथन का कई बार प्रयोग करेंगें,

किसी $x \in C$ एवं $y \in B$ के लिए

$$||x-y|| = H(x,B),\ B \in CL(C).$$

चूंकि $p_n \in Sx_n$ तथा $q_n \in Ty_n$,

$$2||p_n-q_n||$$

$$\le (||x_n-p_n|| + ||y_n-q_n||) + ||x_n-q_n|| + ||y_n--p_n||$$

$$\le H(x_n,Sx_n) + H(y_n,Ty_n) + ||x_n-q_n|| + ||y_n-p_n||.$$

इनसे जैसा कि प्रमेय 1 की उपपत्ति में देखा गया, निम्न सरलता से प्राप्त होते हैं :

$$||x_n-q_n|| \to 0 \qquad ||q_n-z|| \to 0.$$

यदि $x = x_n$ एवं $y=y_n$ के लिए (8a) सन्तुष्ट हो तब

$$2\ ||p_n-q_n||$$

$$\leq a||x_n-y_n|| + ||x_n-q_n| + ||y_n-p_n||$$

$$\leq ab_n(||x_n-q_n|| + ||q_n-p_n||) + ||x_n-q_n||$$

$$+ (1-b_n)\ (||x_n-q_n|| + ||q_n-p_n||)$$

अर्थात्

$$||p_n-q_n||$$

$\leq t||x_n-q_n||$, जहाँ $t = (2 + (a-1)\ b_n)/(1-(a-1)b_n).$

यदि $x = x_n$ एवं $y = y_n$ के लिए (8b) सन्तुष्ट हो, तब

$$2||p_n-q_n||$$

$$\leq b(\ D(x_n,Ty_n) + D(y_n,Sx_n) + ||x_n-y_n|| + (||x_n-q_n||$$

$$+ ||y_n-p_n||)$$

$$\leq (b+1)\ (||x_n-q_n|| + ||y_n-p_n||) + b||x_n-y_n||$$

अर्थात्

$$||p_n-q_n||$$

$\leq t_2\ ||x_n-q_n||, \qquad t_2 = (2 + 2b - b_n)/(1-b+b_n).$

यदि $x = x_n$ एवं $y = y_n$ के लिए (8c) सन्तुष्ट हो, तब

$||p_n-q_n|| \leq t\ H(Sx_n,Ty_n)$, जहाँ $t = k^{-h}$.

अत:

$$(2t + 1)\ ||p_n-q_n|| = t\ ||p_n-q_n|| + t\ ||p_n-q_n||$$

$$+ ||p_n-q_n|| \leq t(||p_n-y_n|| + ||y_n-q_n||)$$

$$+ t(||x_n-p_n|| + ||x_n-q_n|| + tH(Sx_n,Ty_n)$$

$$\leq t\ H(y_n,Ty_n) + tH(x_n, Sx_n) + tH(Sx_n,Ty_n)$$

$$+t(||x_n - q_n|| + ||y_n-p_n||)$$

$$\leq ct\ D(x_n,Ty_n) + D(y_n, Sx_n) + t\ (||x_n-q_n|| + ||y_n-p_n||)$$

$$\leq t(c+1)\ (||x_n-q_n|| + ||y_n-p_n||)$$

अथवा

$$||p_n-q_n|| \leq t_3||x_n-q_n||,$$

जहाँ $t_3=t(c+1)\ (2-b_n)/(1+t(1-c(1-b_n)+b_n))$.

अन्तत: यदि $x = x_n$ एवं $y = y_n$ के लिए (8d) सन्तुष्ट हो, तब

$||p_n-q_n|| \leq$

tk अधिकतम $\{||x_n-y_n||, D(x_n,Sx_n), D(y_n,Ty_n),$
$1/2[D(x_n,Ty_n) + D(y_n,Sx_n)]\}$

$$\leq tk(2||x_n-q_n|| + ||q_n-p_n||).$$

अस्तु,

$||p_n-q_n||$

$\leq t_4||x_n-q_n||$, जहाँ $t_4 = 2tk/(1-tk)$.

इसलिए $x = x_n$ एवं $y = y_n$ के लिए

(9) $||p_n-q_n|| \leq$

अधिकतम $\{t_1,t_2,t_3,t_4\} ||x_n-q_n||$.

स्पष्टतः किसी b_n [0,1] के लिए $t_4 > 0$ एवं t_1, t_2 धनात्मक हैं.

अब t_3 को दृष्टिगत रखते हुए किसी $b_n \in [0,7]$ के लिए t_3 धनात्मक होगा यदि

$1+t(1-c(1-b_n)+b_n) > 0$,

अर्थात् $1+t(1-3(1-b_n)/2+b_n) > 0$ (क्योंकि c व 3/2 आपस में काफी निकट हो सकते हैं).

इसलिए $1+t(1-3/2) > 0$ या $t < 2$.

स्पष्टतया किसी $b_n \in [0,1]$ के लिए $t=2$ होने पर t_3 धनात्मक होगा.
यदि $t > 2$ तब t_3 धनात्मक होगा, यदि

$1+t(1-c(1-b_n)+b_n) > 0$,

अर्थात यदि

$b_n > (c-1)/(t-1)/(tc+t)=(c-1-1/t)/(c+1)$.

इसलिए (8e) को मद्देनज़र रखते हुए t_3 धनात्मक होगा. इस प्रकार (9) से

$||p_n-q_n|| \to 0$, $||x_n - P_n|| \to 0$

एवं $||p_n - z|| \to 0$, जब $n \to \infty$ लें.

अब हम यह दिखायेंगे कि S का स्थिर बिन्दु z है. $x = z$, $y = y_n$ के लिए प्रतिबन्धों (8a)-(8a) का परीक्षण करेंगें. सर्वप्रथम (8a) लेकर

$$\text{(10a)}\quad H(z, Sz) + H(y_n, Ty_n) \le a\|y_n - z\|.$$

इसी तरह (8b) से

$$\begin{aligned}
&\|y_n - q_n\| + H(z,Sz)\\
&\le b(D(y_n,Sz) + D(z,Ty_n) + \|y_n - z\|)\\
&\le b(\|y_n - z\| + D(z,Sz) + \|z-q_n\| + \|y_n - z\|)\\
&\le b(2\|y_n - z\| + \|z-q_n\| + H(z,Sz))
\end{aligned}$$

अर्थात्

$$\text{(10b)}\quad (1-b)\, H(z,Sz) \le 2b\|y_n - z\| - (1-b)\|z-q_n\|.$$

पुनः (8c) से

$$\begin{aligned}
&2H(z,Sz)\\
&\le H(z,Sz) + \|z-y_n\| + H(y_n,Ty_n) + H(Ty_n,Sz)\\
&\le \|z-y_n\| + c(D(y_n,Sz) + D(z,Ty_n))\\
&\le \|z-y_n\| + c(\|y_n - z\| + D(z,Sz) + \|z-q_n\|)\\
&\le (c+1)\|z-y_n\| + cH(z,Sz) + c\|z-q_n\|
\end{aligned}$$

अर्थात्

(10c) $(2-c)\ H(z,Sz)$

$$\le (c+1)\ ||z-y_n|| + c||z-q_n||.$$

अंततः (8d) से

$$D(z,Sz) \le ||z-q_n|| + H(Ty_n,Sz)$$

$$\le ||z-q_n|| + k \text{ अधिकतम } \{||y_n-z||,\ D(y_n,Ty_n),$$

$$D(z,Sz),\ 1/2(D(y_n,z) + D(z,Ty_n))\}$$

(10d) $\le ||z-q_n|| + k$ अधिकतम $\{\ ||y_n-z||,$

$$||y_n-q_n||,\ \ D(z,\ Sz),\ 1/2(||y_n-z|| + D\ (z,Sz)$$

$$+ ||z-q_n||)\}.$$

अतः (10a) - (10c) में सीमा $n \to \infty$ लेने पर

$H(z,Sz) = 0$ एवं (10d) से $D(z,Sz) = 0$ अर्थात् $z \in Sz$.

इसी प्रकार हम सिद्ध कर सकते हैं कि $z \in Tz$.

टिप्पणी 1. प्रमेय 1-2 में $S = T$ लेने पर सिंह [15] के परिणाम उपप्रमेय के रूप में प्राप्त हो जाते हैं.

टिप्पणी 2. प्रमेय 1 में $S=T:\ C \to C$ लेने पर नैम्पल्ली-सिंह [9] की प्रमेय 1.2 को उपप्रमेय के रूप में प्राप्त किया जा सकता है.

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फार्म—४

प्राकृतिक एवं भौतिकीय विज्ञान शोध पत्रिका

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